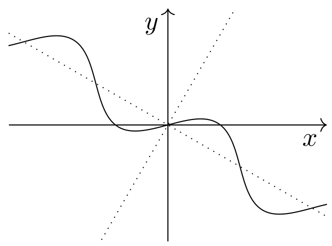
4901. Define new coordinate variables by

$$\begin{aligned} X &= \sqrt{3}x - y, \\ Y &= x + \sqrt{3}y. \end{aligned}$$

The lines $x + \sqrt{3}y = 0$ and $\sqrt{3}x - y$, which give the (X, Y) axes, are perpendicular, and are angled at 30° clockwise to the (x, y) axes.

So, the curve is $y = \sin x$, rotated by 30° clockwise around the origin. It is also enlarged by factor $\frac{1}{2}$, because 2 is the Pythagorean sum of 1 and $\sqrt{3}$:



4902. Let A have coordinates (x, y) and let the angle of projection be ϕ. Quoting the standard equation of the trajectory,

$$y = x\tan\phi - \frac{gx^2}{2u^2}(\tan^2\phi + 1).$$

For the minimum launch speed, this equation, thought of as a quadratic in $\tan\phi$, must have $\Delta = 0$:

$$x^2 + 4\frac{gx^2}{2u^2}\left(\frac{gx^2}{2u^2} + y\right) = 0.$$

Dividing through by x^2,

$$\begin{aligned} & 1 + 4\frac{g}{2u^2}\left(\frac{gx^2}{2u^2} + y\right) = 0 \\ \Longrightarrow\ & u^4 - 2gyu^2 - g^2x^2 = 0 \\ \Longrightarrow\ & u^2 = \frac{2gy \pm \sqrt{4g^2y^2 + 4g^2x^2}}{2} \\ & \quad\equiv \left(y \pm \sqrt{y^2 + x^2}\right). \end{aligned}$$

We reject the $-$ve root, for which $u^2 < 0$, giving

$$u^2 = g\left(y + \sqrt{y^2 + x^2}\right).$$

The solution for $\tan\phi$ is then

$$\begin{aligned} \tan\phi &= \frac{x}{2 \cdot \frac{gx^2}{2u^2}} \\ &\equiv \frac{u^2}{gx} \\ &= \frac{y + \sqrt{y^2 + x^2}}{x} \\ &\equiv \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 1}. \end{aligned}$$

Let the angle between $\overrightarrow{OA}$ and the vertical be 2θ, so that

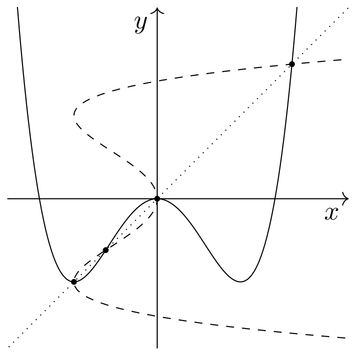
$$\cot 2\theta = \frac{y}{x}.$$

Substituting this in,

$$\begin{aligned} \tan\phi &= \cot 2\theta + \sqrt{\cot^2 2\theta + 1} \\ &= \cot 2\theta + \operatorname{cosec} 2\theta \\ &\equiv \frac{\cos 2\theta + 1}{\sin 2\theta} \\ &\equiv \frac{2\cos^2\theta}{2\sin\theta\cos\theta} \\ &\equiv \cot\theta. \end{aligned}$$

Since $\tan\phi = \cot\theta$, we know that $\theta + \phi = \pi/2$, which puts projection along the angle bisector of $\overrightarrow{OA}$ and the vertical. □

4903. The equations are symmetrical in $y = x$. The SPs of the first curve are at $(0, 0)$ and $(\pm 1, -1)$. Each quartic has even symmetry. So, the graphs, with the line $y = x$, are



The location of the SPs guarantees that all of the intersections lie on the line $y = x$. Solving for these,

$$\begin{aligned} & x^4 - 2x^2 = x \\ \Longrightarrow\ & x(x^3 - 2x - 1) = 0 \\ \Longrightarrow\ & x(x + 1)(x^2 - x - 1) = 0 \\ \Longrightarrow\ & x = 0, -1, \tfrac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

The y values are equal to these.

4904. (a) Since the rate of departure is zero prior to t_0 and subsequently large, we can assume that t_0 is the time at which the event finishes.

Whatever the value of k, the maximum rate of departure is at $t = t_0$; k then controls how quickly the rate drops off. A large value of k means that the rate drops off quickly, so that the crowd will take a long time to disperse.

(b) For the model to be consistent, the total area under the graph must equal the initial size of the crowd, i.e. everyone must leave eventually. Algebraically, this is

$$P_0 = \int_{t_0}^{\infty} \frac{N}{1 + k(t - t_0)^2}\, dt.$$

Let $\tan u = \sqrt{k}(t - t_0)$, so $\sec^2 u \, du = \sqrt{k}\, dt$. The new limits are $u = 0, \frac{\pi}{2}$. This gives

$$\begin{aligned} P_0 &= \frac{N}{\sqrt{k}} \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2 u} \sec^2 u \, du \\ &\equiv \frac{N}{\sqrt{k}} \int_0^{\frac{\pi}{2}} 1 \, du \\ &\equiv \frac{N}{\sqrt{k}} \Big[u \Big]_0^{\frac{\pi}{2}} \\ &\equiv \frac{N\pi}{2\sqrt{k}} \end{aligned}$$

Rearranging this, $N = \dfrac{2P_0\sqrt{k}}{\pi}$, as required.

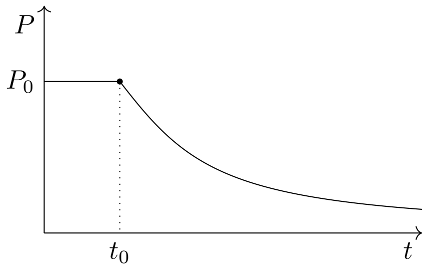
(c) From the above, the integral of $\frac{dP}{dt}$ is

$$-\frac{N}{\sqrt{k}} u + c.$$

The constant of integration is P_0. Using the value of N found above, the number of people present, for $t \geq t_0$, is

$$\begin{aligned} P &= P_0 - \frac{2P_0\sqrt{k}}{\pi\sqrt{k}} u \\ &= P_0\big(1 - \tfrac{2}{\pi} \arctan \sqrt{k}(t - t_0)\big). \end{aligned}$$

The graph of P against t is



4905. The triple-angle identities, which can be proved using compound- and double-angle identities, are

$$\begin{aligned} \sin 3\theta &= 3\sin\theta - 4\sin^3\theta, \\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

So, the LHS is

$$\begin{aligned} &\frac{\sin 3\theta}{\cos\theta} + \frac{\cos 3\theta}{\sin\theta} \\ \equiv\ &\frac{3\sin\theta - 4\sin^3\theta}{\cos\theta} + \frac{4\cos^3\theta - 3\cos\theta}{\sin\theta} \\ \equiv\ &\frac{3\sin^2\theta - 4\sin^4\theta + 4\cos^4\theta - 3\cos^2\theta}{\sin\theta\cos\theta}. \end{aligned}$$

The numerator of the LHS is

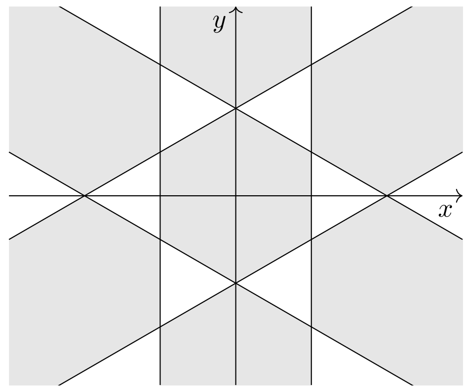
$$\begin{aligned} &3\sin^2\theta(1-\sin^2\theta) - \sin^4\theta \\ &\qquad - 3\cos^2\theta(1-\cos^2\theta) + \cos^4\theta \\ \equiv\ &\cos^4\theta - \sin^4\theta \\ \equiv\ &(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) \\ \equiv\ &\cos 2\theta. \end{aligned}$$

The denominator of the LHS is $\frac{1}{2}\sin 2\theta$. So,

$$\begin{aligned} \text{LHS} &= \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} \\ &\equiv \cot 2\theta, \text{ as required.} \end{aligned}$$

4906. Consider the boundary equation. For the LHS to be zero, at least one of the expressions must be zero. Each expression is a difference of two squares, which gives a set of six lines. These form a stellated regular hexagon.

The origin satisfies the inequality. And crossing any single boundary equation causes a sign change in the LHS. Hence, the successful regions form a chequerboard pattern:



4907. Since $f(p) = 0$, we know that $f(x)$ has a factor of $(x - p)$. Taking this factor out, what remains is some polynomial $f_1(x)$:

$$f(x) = (x-p)\, f_1(x).$$

Differentiating by the product rule,

$$f'(x) = f_1(x) + (x-p)\, f_1'(x).$$

Substituting $x = p$, we know that the LHS is zero and also that the second term on the RHS is zero. So, the first term on the RHS is also zero: $f_1(p) = 0$. Hence, by the factor theorem, $f_1(x)$ has a factor of $(x-p)$. Substituting this back into $f(x) = (x-p)\, f_1(x)$, we now know that $f(x)$ has a factor of $(x-p)^2$.

We can repeat the argument with

$$f''(x) = f_2(x) - (x-p)\, f_2'(x).$$

Substituting $x = p$ shows that $f_2(x)$ has a factor of $(x-p)$, so $f_1(x)$ has a factor of $(x-p)^2$, so $f(x)$ has a factor of $(x-p)^3$.

We apply this argument $n+1$ times, once for f and n times for its derivatives. Each application produces a new factor of $(x-p)$ in $f(x)$. So, $f(x)$ has a factor of $(x-p)^{n+1}$. QED.

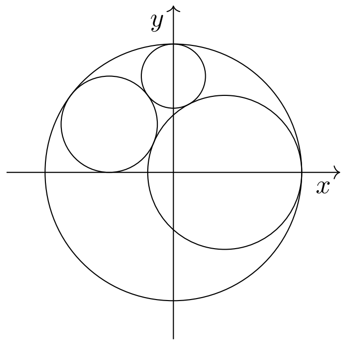
4908. Centre the unit circle at O and the second largest circle at $(2/5, 0)$. We need to find the centre (a, b) of the circle of radius $3/8$. This lies at a distance of $5/8$ from the origin and $3/8 + 3/5 = 39/40$ from $(2/5, 0)$. So,

$$a^2 + b^2 = \tfrac{25}{64},$$
$$\left(a - \tfrac{2}{5}\right)^2 + b^2 = \tfrac{1521}{1600}.$$

Subtracting these,

$$a^2 - \left(a - \tfrac{2}{5}\right)^2 = \tfrac{25}{64} - \tfrac{1521}{1600}$$
$$\Longrightarrow a = -\tfrac{1}{2}.$$

This gives $b = \frac{3}{8}$. The scenario is



Let the centre of the smallest circle be (p, q). This lies at a distance $1 - r$ from the origin, so

$$p^2 + q^2 = (1 - r)^2$$
$$\Longrightarrow p^2 + q^2 = 1 - 2r + r^2. \quad ①$$

The distance from $(2/5, 0)$ gives

$$\left(p - \tfrac{2}{5}\right)^2 + q^2 = \left(\tfrac{3}{5} + r\right)^2,$$
$$\Longrightarrow p^2 - \tfrac{4}{5}p + \tfrac{4}{25} + q^2 = \tfrac{9}{25} + \tfrac{6}{5}r + r^2$$
$$\Longrightarrow p^2 - \tfrac{4}{5}p + q^2 = \tfrac{1}{5} + \tfrac{6}{5}r + r^2. \quad ②$$

The distance from $(-1/2, 3/8)$ gives

$$\left(p + \tfrac{1}{2}\right)^2 + \left(q - \tfrac{3}{8}\right)^2 = \left(\tfrac{3}{8} + r\right)^2. \quad ③$$

Subtracting ② from ①,

$$\tfrac{4}{5}p = \tfrac{4}{5} - \tfrac{16}{5}r$$
$$\Longrightarrow p = 1 - 4r.$$

Substituting this into ①,

$$(1 - 4r)^2 + q^2 = 1 - 2r + r^2$$
$$\Longrightarrow 1 - 8r + 16r^2 + q^2 = 1 - 2r + r^2$$
$$\Longrightarrow q^2 = 6r - 15r^2$$
$$\therefore \quad q = \sqrt{6r - 15r^2}.$$

And the same into ③,

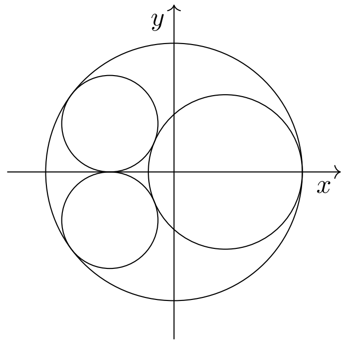
$$\left(\tfrac{3}{2} - 4r\right)^2 + \left(\sqrt{6r - 15r^2} - \tfrac{3}{8}\right)^2 = \left(\tfrac{3}{8} + r\right)^2$$
$$\Longrightarrow 9r + \sqrt{6r - 15r^2} = 3$$
$$\Longrightarrow r = \tfrac{1}{4}.$$

— Nota Bene —

This is a case of *Descartes' theorem*. The theorem states that, in this problem, the radii are linked by

$$\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r^4}\right)^2 = 2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right)$$

With $r_2 = \frac{3}{8}, r_3 = \frac{3}{5}$ and $r_4 = 1$, we get $r_1 = \frac{1}{4}$ or $\frac{3}{8}$. The former solves the problem as set. The latter corresponds to the following scenario:



4909. Writing this longhand, we have

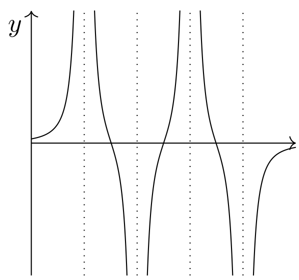
$$\frac{1}{(x-1)^2} - \frac{1}{(x-2)^2} + \frac{1}{(x-3)^2} - \ldots + (-1)^{n+1}\frac{1}{(x-n)^2} = 0.$$

Consider the graph $y = \text{LHS}$. Each of the terms generates a double asymptote at $x = r$. So, there are n double asymptotes at $x = 1, 2, ..., n$.

Consider the graph near an asymptote: $x \to r^{\pm}$. In this limit, the term generating the asymptote at $x = r$ dominates all others. Hence,

- if r is odd, $y \to \infty$,
- if r is even, $y \to -\infty$.

So, between each successive vertical asymptote, the y value of the graph must change sign. Hence, since there are no other discontinuities, there must be at least one root between each successive pair of asymptotes.



Since there are n asymptotes, there must be at least $n - 1$ roots, as required.

4910. (a) Place A and B at $(\mp 0.3, 0)$, and place the ring at (x, y). The total distance to A and B is

$$d = \sqrt{(x+0.3)^2 + y^2} + \sqrt{(x-0.3)^2 + y^2}.$$

On the ellipse,

$$y^2 = \tfrac{4}{25} - \tfrac{16}{25}x^2.$$

Substituting this in,

$$\begin{aligned} d &= \sqrt{\tfrac{1}{100}(5+6x)^2} + \sqrt{\tfrac{1}{100}(5-6x)^2} \\ &= \tfrac{1}{10}(5+6x) + \tfrac{1}{10}(5-6x) \\ &= 1. \end{aligned}$$

So, on the ellipse, the total distance to A and B is 1. This ellipse is, therefore, the location of a ring threaded on a taut string of length 1.

(b) The ring's velocity must be tangential to the ellipse. To find this direction, we differentiate implicitly:

$$\begin{aligned} 32x + 50y\frac{dy}{dx} &= 0 \\ \Longrightarrow \frac{dy}{dx} &= -\frac{16x}{25y}. \end{aligned}$$

At the initial position, which is $(-0.3, -0.32)$, the gradient of the ellipse is

$$m = -\frac{16 \cdot -0.3}{25 \cdot -0.32} = -\frac{3}{5}.$$

So, the angle of inclination of the tangent in the initial position is $\alpha = \arctan \frac{3}{5}$. Giving this as a direction in the usual sense (in this case anticlockwise from the horizontal from A to B), it is $-\alpha$ or $2\pi - \alpha$. From this value, the direction goes to zero at the nadir, and then to α just prior to turning below B. After the turn below B, the direction is $\pi + \alpha$. This goes to π at the nadir, and then to $\pi - \alpha$ just before return to the initial position. The α boundaries are not attainable, because the velocity drops to zero as the ring turns, rendering direction meaningless. So, with α defined as above, the set of possible values of the direction of the velocity is

$$[0, \alpha) \cup (\pi - \alpha, \pi + \alpha) \cup (2\pi - \alpha, 2\pi).$$

4911. Let $u = \ln x$. This gives $du = \frac{1}{x}\, dx$, which can be written $dx = e^u\, du$. Enacting the substitution,

$$\int \sin(\ln x)\, dx = \int e^u \sin u \, du.$$

Call this I. For the tabular integration method,

Signs	Derivatives	Integrals
$+$	e^u	$\sin u$
$-$	e^u	$-\cos u$
$+$	e^u	$-\sin u$

This gives

$$\begin{aligned} I &= -e^u \cos u + e^u \sin u - I \\ \Longrightarrow I &= \tfrac{1}{2}e^u(\sin u - \cos u). \end{aligned}$$

So, the full result is

$$\begin{aligned} &\int \sin(\ln x)\, dx \\ &= \tfrac{1}{2}e^{\ln x}\big(\sin(\ln x) - \cos(\ln x)\big) + c \\ &= \tfrac{1}{2}x\big(\sin(\ln x) - \cos(\ln x)\big) + c. \end{aligned}$$

4912. We need a and b such that

$$\big(a + b\sqrt{2}\big)^3 = 38752 + 28310\sqrt{2}.$$

Expanding and equating coefficients, this is

$$\begin{aligned} a^3 + 6ab^2 &= 38752, \\ 3a^2b + 2b^3 &= 28310. \end{aligned}$$

Rearranging the second equation,

$$a = \sqrt{\frac{28310 - 2b^3}{3b}}.$$

So, we need to solve

$$\left(\frac{28310 - 2b^3}{3b}\right)^{\frac{3}{2}} + 6\left(\frac{28310 - 2b^3}{3b}\right)^{\frac{1}{2}} b^2 = 38752.$$

Multiplying by $(3b)^{\frac{3}{2}}$,

$$\begin{aligned} &\big(28310 - 2b^3\big)^{\frac{3}{2}} + 18\big(28310 - 2b^3\big)^{\frac{1}{2}} b^3 \\ &\qquad - 38752(3b)^{\frac{3}{2}} = 0. \end{aligned}$$

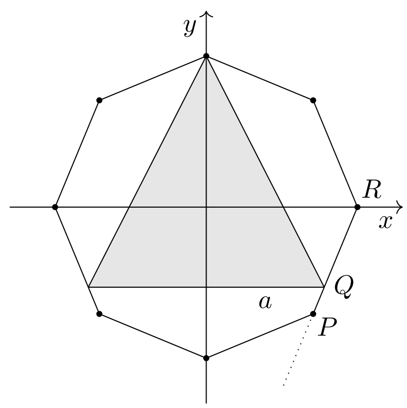
We are looking for an integer solution, so can use a sign change method. Notating the above $\mathrm{f}(b) = 0$, we evaluate at intervals of 10. The expression $\mathrm{f}(b)$ is undefined for $b < 0$ and for $b \geq 30$.

b	0	10	20
$\mathrm{f}(b)$	4.8×10^6	8.2×10^5	-6.7×10^5

Checking the integers between 10 and 20, we see that $\mathrm{f}(19) = 0$. This, in turn, gives $a = -16$. So,

$$-16 + 19\sqrt{2} = \sqrt[3]{38752 + 28310\sqrt{2}}.$$

4913. Setting up x and y axis and labelling a length and various points, the scenario is



Let the octagon have radius 1, so that R is at $(1, 0)$. The gradient of PR is $\tan 67.5° = 1 + \sqrt{2}$. So, the equation of line PR is

$$y = \big(1 + \sqrt{2}\big)(x - 1).$$

The coordinates of Q are

$$\Big(a, \big(1 + \sqrt{2}\big)(a - 1)\Big).$$

The height of the triangle is therefore

$$\begin{aligned} h &= 1 - \big(1 + \sqrt{2}\big)(a - 1) \\ &\equiv 2 + \sqrt{2} - \big(1 + \sqrt{2}\big)a. \end{aligned}$$

So, the area of the triangle is

$$A_\triangle = \big(2 + \sqrt{2}\big)a - \big(1 + \sqrt{2}\big)a^2.$$

To optimise this, we set the derivative to zero:

$$\begin{aligned} \frac{dA}{da} &= \big(2 + \sqrt{2}\big) - \big(2 + 2\sqrt{2}\big)a = 0 \\ \implies a &= \tfrac{\sqrt{2}}{2}. \end{aligned}$$

This is point P. We have proved that, anywhere on the line PR (including on the dotted extension in the diagram), the area is maximised at P. Hence, we have no need of checking vertices on the edge between $(0, -1)$ and P. The area of the triangle is maximised when its vertices coincide with those of the octagon.

— Nota Bene —

The value $\tan 67.5° = 1 + \sqrt{2}$ can be derived from the double-angle formula

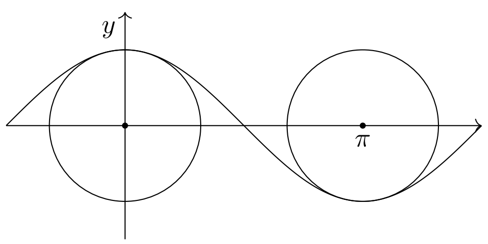
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Set $\theta = 67.5°$. Then $\tan 2\theta = \tan 135° = -1$. This gives

$$\begin{aligned} &\tan^2 67.5° - 1 = 2\tan 67.5° \\ \implies &\tan^2 67.5° - 2\tan 67.5° - 1 = 0. \end{aligned}$$

Taking the positive root in the quadratic formula gives $\tan 67.5° = 1 + \sqrt{2}$.

4914. At $x = 0$, the curve $y = \cos x$ can be approximated with a unit circle. Hence, the radius of curvature at $x = 0$ is 1. The same applies at $x = \pi$.



4915. At the point of self-intersection,

$$\begin{aligned} \ln s \cos s &= \ln t \cos t, \\ -\ln s \sin s &= -\ln t \sin t. \end{aligned}$$

Dividing these equations, $\tan s = \tan t$. We know that $s \neq t$, so, assuming s to be the smaller, $s + \pi = t$. Substituting this in,

$$\begin{aligned} &\ln s \cos s = \ln(s + \pi)(\cos s + \pi) \\ \implies &\ln s = -\ln(s + \pi) \\ \implies &\ln s + \ln(s + \pi) = 0 \\ \implies &\ln\big(s(s + \pi)\big) = 0 \\ \implies &s^2 + \pi s - 1 = 0 \\ \implies &s = \frac{-\pi \pm \sqrt{\pi^2 + 4}}{2}. \end{aligned}$$

The negative root gives $s \notin [0, 2\pi)$. So,

$$s = \frac{-\pi + \sqrt{\pi^2 + 4}}{2}.$$

Adding π to this, the parameters are

$$s, t = \frac{\sqrt{\pi^2 + 4} \mp \pi}{2}, \text{ as required.}$$

4916. The second derivative $\text{h}''(x)$ is a quadratic. Since it has roots at $x = a$ and $x = b$, it is symmetrical around the midpoint of the two. So, the following holds for all $x \in \mathbb{R}$:

$$\text{h}''\left(\frac{a + b}{2} - x\right) = \text{h}''\left(\frac{a + b}{2} + x\right).$$

We integrate this by the reverse chain rule, using a single constant of integration on the RHS:

$$-\text{h}'\left(\frac{a + b}{2} - x\right) = \text{h}'\left(\frac{a + b}{2} + x\right) + c.$$

Substituting $x = \dfrac{b - a}{2}$,

$$-\text{h}'(a) = \text{h}'(b) + c.$$

We are told that $\text{h}'(a) + \text{h}'(b) = 0$, so the constant $c = 0$. Integrating again, the following holds for all $x \in \mathbb{R}$:

$$\text{h}\left(\frac{a + b}{2} - x\right) = \text{h}\left(\frac{a + b}{2} + x\right) + d.$$

Using the same substitution, $d = 0$. Hence, the required result holds for all $x \in \mathbb{R}$:

$$\text{h}\left(\frac{a + b}{2} - x\right) = \text{h}\left(\frac{a + b}{2} + x\right).$$

4917. The tangent at $x = p$ is $y = 2px - p^2$. Setting $y = 0$, we get $2px = p^2$, so the x axis intercept Q of the tangent is at $x = \frac{p}{2}$. Calculating the squared distance to P:

$$|PQ|^2 = \left(p - \tfrac{p}{2}\right)^2 + p^4 = \tfrac{1}{4}p^2 + p^4.$$

Since tangents to a point are the same length, this is equal to $\left(1 - \frac{p}{2}\right)^2$. So,

$$\begin{aligned}&\tfrac{1}{4}p^2 + p^4 = 1 - p + \tfrac{1}{4}p^2\\ \Longrightarrow\ &p^4 + p - 1 = 0.\end{aligned}$$

The Newton-Raphson iteration is

$$p_{n+1} = p_n - \frac{p_n^4 + p_n - 1}{4p_n^3 + 1}.$$

Running this with $p_0 = 1$, we get $p_1 = 0.8$, then $p_n \to 0.72449$. This suggests $p = 0.7245$ to 4sf. Verifying the root with a sign change, we define $\text{f}(p) = p^4 + p - 1$:

$$\begin{aligned}\text{f}(0.72445) &= -0.000105... < 0,\\ \text{f}(0.72455) &= 0.000146 > 0.\end{aligned}$$

So, we can confirm $p = 0.7245$ to 4sf.

4918. Parametrising the unit circle with the usual angle θ, the average value A is given by

$$A = \tfrac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} \cos^2\theta \sin^2\theta \, d\theta.$$

Using two double-angle formulae,

$$\begin{aligned}A &= \tfrac{1}{8\pi}\int_0^{2\pi} \sin^2 2\theta \, d\theta\\ &= \tfrac{1}{16\pi}\int_0^{2\pi} 1 - \cos 4\theta \, d\theta\\ &= \tfrac{1}{16\pi}\Big[\theta - \tfrac{1}{4}\sin 4\theta\Big]_0^{2\pi}\\ &= \tfrac{1}{16\pi} \times 2\pi\\ &= \tfrac{1}{8}, \text{ as required.}\end{aligned}$$

4919. (a) Taking out a factor of $\dfrac{1}{n^2}$,

$$I = \frac{1}{n^2}\int nx^{n-1} \cdot n \ln x \, dx.$$

Then, using a log rule,

$$I = \frac{1}{n^2}\int nx^{n-1} \ln(x^n) \, dx.$$

(b) Since nx^{n-1} is the derivative of the inside x^n, we can integrate by inspection (by the reverse chain rule). The result is $\text{F}(x^n) + c$, in which $\text{F}(x) = x \ln x - x$:

$$\begin{aligned}I &= \frac{1}{n^2}\big(x^n \ln(x^n) - x^n\big) + c\\ &\equiv \frac{x^n}{n^2}\big(\ln(x^n) - 1\big) + c\\ &\equiv \frac{x^n}{n^2}(n \ln x - 1) + c, \text{ as required.}\end{aligned}$$

4920. The equation of the normal at $x = p$ is

$$\begin{aligned}&y - p^2 = -\tfrac{1}{2p}(x - p)\\ \Longrightarrow\ &y = -\tfrac{1}{2p}x + p^2 + \tfrac{1}{2}.\end{aligned}$$

For intersections with the curve,

$$\begin{aligned}&x^2 = -\tfrac{1}{2p}x + p^2 + \tfrac{1}{2}\\ \Longrightarrow\ &x^2 + \tfrac{1}{2p}x - p^2 - \tfrac{1}{2} = 0\\ \Longrightarrow\ &2px^2 + x - 2p^3 - p = 0.\end{aligned}$$

Factorising,

$$(x - p)(2px + 2p^2 + 1) = 0.$$

The first factor is the original point $\left(p, p^2\right)$. The latter gives the re-intersection at

$$\begin{aligned}&2px + 2p^2 + 1 = 0\\ \Longrightarrow\ &x = p + \tfrac{1}{2p}.\end{aligned}$$

At this point, the y coordinate is

$$\begin{aligned}y &= \left(p + \tfrac{1}{2p}\right)^2\\ &\equiv p^2 + 1 + \frac{1}{4p^2}.\end{aligned}$$

For the range of this expression, we look for SPs:

$$\begin{aligned}&2p - \tfrac{1}{2p^3} = 0\\ \Longrightarrow\ &p = \pm\tfrac{\sqrt{2}}{2}.\end{aligned}$$

All terms are positive in $y = p^2 + 1 + \frac{1}{4p^2}$, so these must be global minima. The y value at these SPs is $y = 2$. Hence, the normal re-intersects the curve at $y \geq 2$, as required.

4921. We multiply top/bottom by $\left(1 - 2^{\frac{1}{4}} + 2^{\frac{1}{2}} - 2^{\frac{3}{4}}\right)$. This gives

$$\begin{aligned}&\frac{1}{1 + \sqrt[4]{2}}\\ =\ &\frac{\left(1 - 2^{\frac{1}{4}} + 2^{\frac{1}{2}} - 2^{\frac{3}{4}}\right)}{\left(1 + 2^{\frac{1}{4}}\right)\left(1 - 2^{\frac{1}{4}} + 2^{\frac{1}{2}} - 2^{\frac{3}{4}}\right)}\\ =\ &\frac{1 - 2^{\frac{1}{4}} + 2^{\frac{1}{2}} - 2^{\frac{3}{4}}}{-1}\\ =\ &2^{\frac{3}{4}} - 2^{\frac{1}{2}} + 2^{\frac{1}{4}} - 1.\end{aligned}$$

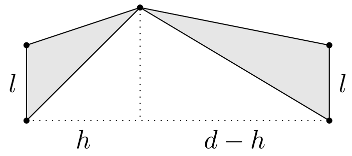
— Nota Bene —

The choice as to what to multiply by here is due to the product/factorisation

$$(1 + x)\left(1 - x + x^2 - x^3\right) \equiv 1 - x^4.$$

Equivalent results allow for rationalisation of the denominator of $\left(a + b\sqrt[n]{c}\right)^{-1}$ for any $n \in \mathbb{N}$.

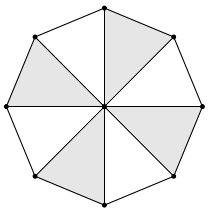
4922. Since $2n$ is even, each triangle is paired with one opposite it. The bases of such a pair are parallel lines of the same length l; the distance between them is the midpoint-to-midpoint diameter of the polygon d:



The total area shaded is

$$\begin{aligned} A_{\text{pair}} &= \tfrac{1}{2}lh + \tfrac{1}{2}l(d-h) \\ &\equiv \tfrac{1}{2}ld. \end{aligned}$$

This is independent of the position of the central point. So, moving it does not affect the total area shaded. Consider moving it to the middle:



The result clearly holds above. So, irrespective of the positioning of the central point, half of the polygon is shaded. QED.

4923. The probability that there is at least one heart is

$$1 - \frac{{}^{39}\mathrm{C}_4}{{}^{52}\mathrm{C}_4} = 0.69618...$$

Hence, the expectation of the *presence* of a heart in the hand, with 1 as presence and 0 as absence, is 0.69618. This is the same for all the suits. Hence, the total expectation is $4 \times 0.69618 \approx 2.785$.

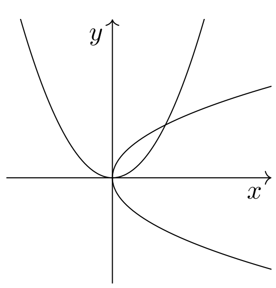
— Nota Bene —

This can be verified by brute force, calculating the probabilities of $\{1, 2, 3, 4\}$ suits present. This is a good but long and tricky exercise in probability.

4924. The relation factorises as follows:

$$\begin{aligned} & x^3 - x^2y^2 - xy + y^3 = 0 \\ \Longrightarrow\ & \left(x^2 - y\right)\left(x - y^2\right) = 0. \end{aligned}$$

So, the locus of points is a pair of symmetrical intersecting parabolae: $y = x^2$ and $x = y^2$.

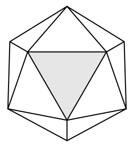


4925. There are 20! ways in which the numbers can be placed. An icosahedron has 12 vertices. For each choice of vertex, there are 5! ways of arranging the numbers $1, 2, 3, 4, 5$ around it. There are then 15! ways of arranging the other 15 numbers. This gives the probability as

$$p = \frac{12 \times 5! \times 15!}{20!} = \tfrac{1}{1292}.$$

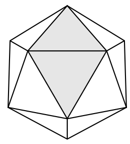
— Alternative Method —

Place 1 wlog:



Consider the placement of 2. There are nine faces which share a vertex with 1, of which three share an edge. So, there are two cases, with probability $\frac{3}{19}$ and $\frac{6}{19}$:

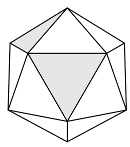
① 1 and 2 share an edge.



In this case, there remain two vertices to choose from. The probability of success via this route is

$$p_1 = 1 \cdot \tfrac{3}{19} \cdot \tfrac{6}{18} \cdot \tfrac{2}{17} \cdot \tfrac{1}{16} = \tfrac{1}{2584}.$$

② 1 and 2 share a vertex, but no edge:



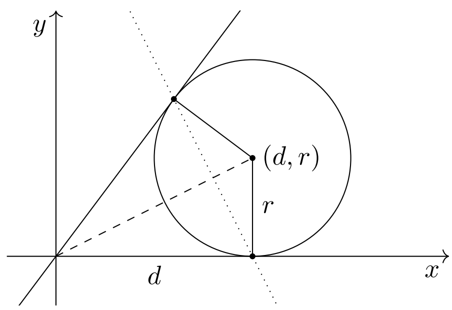
In this case, the vertex is chosen. So, the probability of success via this route is

$$p_2 = 1 \cdot \tfrac{6}{19} \cdot \tfrac{3}{18} \cdot \tfrac{2}{17} \cdot \tfrac{1}{16} = \tfrac{1}{2584}.$$

The total probability of success is

$$p_1 + p_2 = \tfrac{2}{2584} = \tfrac{1}{1292}.$$

4926. (a) The situation, with subsequent lines added, is



(b) The dashed line L_1 has equation $y = \frac{r}{d}x$.

(c) The dotted line L_2 has equation $y = -\frac{d}{r}(x-d)$.

(d) Substituting for y,

$$(x-d)^2 + \left(-\tfrac{d}{r}(x-d) - r\right)^2 - r^2 = 0.$$

Let $X = x - d$. Getting rid of the minus signs in the second term (they are squared),

$$\begin{aligned} X^2 + \left(\frac{d}{r}X + r\right)^2 - r^2 &= 0 \\ \Longrightarrow X^2 + \frac{d^2}{r^2}X^2 + 2dX &= 0 \\ \Longrightarrow \left(1 + \frac{d^2}{r^2}\right)X^2 + 2dX &= 0. \end{aligned}$$

So, either $X = 0$, which is the original point of intersection $(d, 0)$, or

$$\begin{aligned} \left(1 + \frac{d^2}{r^2}\right)X + 2d &= 0 \\ \Longrightarrow X &= -\frac{2d}{1 + \frac{d^2}{r^2}} \\ &\equiv -\frac{2dr^2}{d^2 + r^2}. \end{aligned}$$

The x coordinate of the second point is

$$\begin{aligned} x &= d - \frac{2dr^2}{d^2 + r^2} \\ &\equiv \frac{d(d^2 - r^2)}{d^2 + r^2}. \end{aligned}$$

On L_2, the corresponding y coordinate is

$$\begin{aligned} y &= -\frac{d}{r} \cdot -\frac{2dr^2}{d^2 + r^2} \\ &\equiv \frac{2d^2 r}{d^2 + r^2}. \end{aligned}$$

The coordinates are $\left(\dfrac{d(d^2 - r^2)}{d^2 + r^2}, \dfrac{2d^2 r}{d^2 + r^2}\right)$.

(e) The distance to the origin is

$$\begin{aligned} d_1 &= \frac{d}{d^2 + r^2}\sqrt{(d^2 - r^2)^2 + 4d^2r^2} \\ &\equiv \frac{d}{d^2 + r^2}\sqrt{d^4 - 2d^2r^2 + r^4 + 2d^2r^2} \\ &\equiv \frac{d}{d^2 + r^2}\sqrt{d^4 + 2d^2r^2 + r^4} \\ &\equiv \frac{d}{d^2 + r^2}\sqrt{(d^2 + r^2)^2} \\ &\equiv d. \end{aligned}$$

So, the lengths of the tangents from a point to a circle are the same. □

— Nota Bene —

This is certainly not the easiest way to prove this result, which is obvious by symmetry/congruency of triangles. However, it is often a good exercise to prove an obvious result by non-obvious means, not least because it is rather satisfying when, things having started to look rather complicated, they then go ping!

4927. (a) Differentiating by the product rule,

$$\frac{dx}{dt} = \sin t + t\cos t.$$

The parametric integration formula is

$$A = \int_{t_1}^{t_2} y\frac{dx}{dt}\, dt.$$

In this instance,

$$\begin{aligned} A &= \int_0^{2\pi} t\cos t(\sin t + t\cos t)\, dt \\ &= \tfrac{1}{2}\int_0^{2\pi} 2t\sin t\cos t + 2t^2\cos^2 t\, dt \\ &= \tfrac{1}{2}\int_0^{2\pi} t\sin 2t + t^2(\cos 2t + 1)\, dt \\ &= \tfrac{1}{2}\int_0^{2\pi} t^2 + t^2\cos 2t + t\sin 2t\, dt. \end{aligned}$$

— Nota Bene —

The integral calculates the area directly (all regions contributing positively as opposed to negatively in places), because, at e.g. $t = \pi$, which is the negative y intercept, both y and $\frac{dx}{dt}$ are negative.

(b) Using the tabular integration method,

Signs	Derivatives	Integrals
$+$	t^2	$\cos 2t$
$-$	$2t$	$\frac{1}{2}\sin 2t$
$+$	2	$-\frac{1}{4}\cos 2t$
$-$	0	$-\frac{1}{8}\sin 2t.$

So, the integral of $t^2\cos 2t$ is

$$\left(\tfrac{1}{2}t^2 - \tfrac{1}{4}\right)\sin 2t + \tfrac{1}{2}t\cos 2t.$$

The integral of $t\sin 2t$ is

$$\tfrac{1}{4}\sin 2t - \tfrac{1}{2}\cos 2t.$$

The contributions to the definite integral are

① $\left[\tfrac{1}{3}t^3\right]_0^{2\pi} = \tfrac{8}{3}\pi^3$.

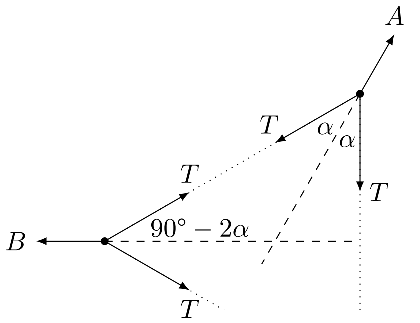
② $\left[\left(\tfrac{1}{2}t^2 - \tfrac{1}{4}\right)\sin 2t + \tfrac{1}{2}t\cos 2t\right]_0^{2\pi} = 0$.

③ $\left[\tfrac{1}{4}\sin 2t - \tfrac{1}{2}\cos 2t\right]_0^{2\pi} = 0$.

Together with the original $1/2$,

$$\begin{aligned} A &= \tfrac{1}{2}\cdot\left(\tfrac{8}{3}\pi^3 + 0 + 0\right) \\ &= \tfrac{4}{3}\pi^3. \end{aligned}$$

4928. The upper half of the loop is



We resolve along the bisectors. The LH vertex gives

$$\begin{aligned} B &= 2T\cos(90° - 2\alpha) \\ &\equiv 2T\sin 2\alpha \\ &\equiv 4T\sin\alpha\cos\alpha. \end{aligned}$$

The other vertex gives $A = 2T\cos\alpha$. Substituting for $2T\cos\alpha$, we get $B = 2A\sin\alpha$. Squaring this,

$$\begin{aligned} B^2 &= 4A^2\sin^2\alpha \\ &\equiv 4A^2(1-\cos^2\alpha) \\ &= 4A^2\left(1 - \frac{A^2}{4T^2}\right). \end{aligned}$$

We then rearrange to make T^2 the subject:

$$T^2 = \frac{A^4}{4A^2 - B^2}.$$

Since $A, B, T > 0$ and $4A^2 > B^2$, we take the positive square root, which gives

$$T = \frac{A^2}{\sqrt{4A^2 - B^2}}, \text{ as required.}$$

4929. The number n may or may not be a perfect square. We consider these case by case:

① Suppose n is not a perfect square. There is no p such that $p^2 = n$, so every divisor must appear as part of a pair: $pq = n$, where $p \neq q$. Hence, $\text{g}(n)$ is even.

② Suppose n is a perfect square, with $k^2 = n$. Every divisor appears in a pair, except k, which is paired with itself. So, $\text{g}(n)$ is one more than an even number, and is odd.

Hence, $\text{g}(n)$ is odd iff n is a perfect square. QED.

4930. The parametric equations of the ellipse are

$$\begin{aligned} x &= a\cos t, \\ y &= b\sin t. \end{aligned}$$

So, the area of the triangle is given by

$$\begin{aligned} A_\triangle &= b\sin t(a - a\cos t) \\ &\equiv ab\sin t - ab\sin t\cos t \\ &\equiv \tfrac{1}{2}ab(2\sin t - \sin 2t). \end{aligned}$$

Setting the derivative to zero,

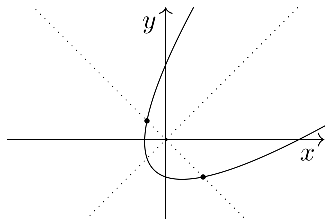
$$\begin{aligned} & 2\cos t - 2\cos 2t = 0 \\ \Longrightarrow\ & 2\cos^2 t - \cos t - 1 = 0 \\ \Longrightarrow\ & (2\cos t + 1)(\cos t - 1) = 0 \\ \Longrightarrow\ & \cos t = 1, -\tfrac{1}{2}. \end{aligned}$$

The former is a minimum, with $A_\triangle = 0$. At the latter, $t = 2\pi/3$. This is a maximum. So, the area of the triangle satisfies

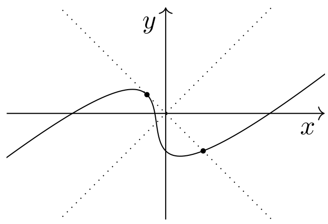
$$\begin{aligned} A_\triangle &\leq \tfrac{1}{2}ab\left(\sqrt{3} - \left(-\tfrac{\sqrt{3}}{2}\right)\right) \\ &\equiv \tfrac{3\sqrt{3}}{4}ab, \text{ as required.} \end{aligned}$$

4931. Let $q = x + y$ and $p = x - y$.

(a) The curve is $q = (p - a)(p - b)$. In the (p, q) plane, this is a positive quadratic with two roots at $a < b$. The (p, q) axes are angled at 45° to the (x, y) axes:



(b) The curve is $q = (p - a)^2(p - b)$. In the (p, q) plane, this is a positive cubic with a double root at $p = a$ and a single root at $p = b$:



4932. The reciprocal triangle numbers are

$$\frac{1}{T_n} = \frac{2}{n(n+1)}.$$

Writing this in partial fractions,

$$\frac{1}{T_n} = \frac{2}{n} - \frac{2}{n+1}.$$

So, the sum of the reciprocal triangle numbers is

$$S_\infty = \tfrac{2}{1} - \tfrac{2}{2} + \tfrac{2}{2} - \tfrac{2}{3} + \tfrac{2}{3} - \tfrac{2}{4} + \ldots$$

We know that the sum converges, so we can group the terms as follows:

$$S_\infty = \tfrac{2}{1} + \left(-\tfrac{2}{2} + \tfrac{2}{2}\right) + \left(-\tfrac{2}{3} + \tfrac{2}{3}\right) + \ldots$$

So, $S_\infty = 2$, as required.

——— Nota Bene ———

In an infinite sum, if the sum doesn't converge, you can't group terms as above. The classic example is *Grandi's series*:

$$S = 1 - 1 + 1 - 1 + 1 - 1 + ...$$

It doesn't converge. And, if you group the terms in different ways, you get contradictory results:

$$\begin{aligned} S &= (1-1) + (1-1) + (1-1) + ... = 0, \\ S &= 1 + (-1+1) + (-1+1) + ... = 1. \end{aligned}$$

Neither is valid. You can't manipulate a sum which doesn't exist in the first place!

4933. The other two angles of inclination, measured in the same sense as θ, are $\theta + 45°$ and $\theta + 90°$. Around the circumference, working with the whole system, the tensions cancel. At angle ϕ, the component of weight is $W \cos\phi$. The direction is set by the sign of $\cos\phi$ itself, hence all terms start with plus signs. The resultant force is

$$2mg\cos\theta + mg\cos(\theta + 45°) + mg\cos(\theta + 90°).$$

We can ignore the factor of mg. Using the identity $\cos(\theta + 90°) \equiv -\sin\theta$, the relevant quantity is

$$2\cos\theta + \cos(\theta + 45°) - \sin\theta.$$

Using a compound-angle formula, this is

$$\begin{aligned} &2\cos\theta + \tfrac{\sqrt{2}}{2}\cos\theta - \tfrac{\sqrt{2}}{2}\sin\theta - \sin\theta \\ \equiv\ &\tfrac{4+\sqrt{2}}{2}\cos\theta - \tfrac{2+\sqrt{2}}{2}\sin\theta. \end{aligned}$$

Setting to zero for equilibrium,

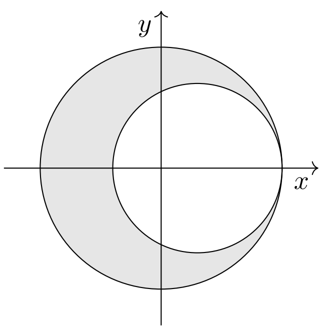
$$\begin{aligned} &\left(4+\sqrt{2}\right)\cos\theta = \left(2+\sqrt{2}\right)\sin\theta \\ \Longrightarrow\ &\tan\theta = \tfrac{4+\sqrt{2}}{2+\sqrt{2}} \\ & = 3 - \sqrt{2}, \text{ as required.} \end{aligned}$$

4934. The boundary equations are formed by setting each factor to zero.

① The first boundary equation is a unit circle centred on the origin.

② The second boundary equation is a circle of radius $k < 1$, centre $(1-k, 0)$. This is inside the unit circle and tangent to it at $(1, 0)$.

The points which satisfy the inequality are those where one factor is positive and the other negative. This is all points within the unit circle and outside the other:



4935. Quoting a standard result, the trajectory before the first bounce is

$$y = -\frac{gx^2}{2u^2} + c.$$

Setting y to zero, the first bounce occurs at

$$x = u\sqrt{\tfrac{2c}{g}}.$$

So, each new trajectory is translated by twice this, which is

$$\delta x = u\sqrt{\tfrac{8c}{g}}.$$

For the trajectory between the nth and $(n+1)$th bounces, we translate the original one by

$$\Delta x = nu\sqrt{\tfrac{8c}{g}}.$$

To enact this translation, we replace x in the first equation, giving

$$\begin{aligned} y &= -\frac{g\left(x - nu\sqrt{\frac{8c}{g}}\right)^2}{2u^2} + c \\ &\equiv -\frac{g\left(x^2 - 2nu\sqrt{\frac{8c}{g}}x + n^2u^2\frac{8c}{g}\right)}{2u^2} + c \\ &\equiv -\frac{gx^2}{2u^2} + \frac{nx}{u}\sqrt{8cg} - 4cn^2 + c \\ &\equiv -\frac{gx^2}{2u^2} + \frac{nx}{u}\sqrt{8cg} - c(4n^2 - 1), \text{ as required.} \end{aligned}$$

4936. (a) Let $x = 2t$, so that $dx = 2\,dt$. This gives

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \frac{4\cos 2t}{(1+\cos 2t)^3} \cdot 2\,dt \\ =\ &\int_0^{\frac{\pi}{4}} \frac{16\cos^2 t - 8}{(2\cos^2 t)^3}\,dt \\ =\ &\int_0^{\frac{\pi}{4}} 2\sec^4 t - \sec^6 t\,dt. \end{aligned}$$

(b) Taking out a factor of $\sec^2 t$, we use the second Pythagorean trig identity:

$$\begin{aligned} &\int_0^{\frac{\pi}{4}} \sec^2 t(2\sec^2 t - \sec^4 t)\,dt \\ =\ &\int_0^{\frac{\pi}{4}} \sec^2 t(2 + 2\tan^2 t - (1+\tan^2 t)^2\,dt \\ =\ &\int_0^{\frac{\pi}{4}} \sec^2 t(1 - \tan^4 t)\,dt. \end{aligned}$$

Integrating by inspection, this is

$$\Big[\tan t - \tfrac{1}{5}\tan^5 t\Big]_0^{\frac{\pi}{4}} = \tfrac{4}{5}.$$

Scaling by $\frac{5}{4}$, the value of the required integral is $\frac{4}{5} \times \frac{5}{4} = 1$.

4937. The factorisation $N = 2^{p-1}(2^p - 1)$ is, in fact, a prime factorisation, since 2 is prime and we know that the latter factor $(2^p - 1)$ is prime. Hence, the factors of N come in two sets. Firstly, the powers of two: $\{1, 2, ..., 2^{p-1}\}$. Secondly, these powers of two multiplied by $(2^p - 1)$, with the exception of N itself: $\{(2^p - 1), 2(2^p - 1), ..., 2^{p-2}(2^p - 1)\}$.

The sums of these are geometric series. The first has $a = 1$, $r = 2$, $n = p$; the second has $a = 2^p - 1$, $r = 2$, $n = p - 1$. So, using the standard formula, the sum of the factors is

$$\begin{aligned} S &= \frac{1(1 - 2^p)}{1 - 2} + \frac{(2^p - 1)(1 - 2^{p-1})}{1 - 2} \\ &\equiv (2^p - 1)(1 + 2^{p-1} - 1) \\ &\equiv (2^p - 1)2^{p-1} \\ &= N. \end{aligned}$$

Hence, N is perfect, as required.

4938. The boundary equation is $x^2 + y^2 = 8$. We solve $xy(x+y) = 16$ simultaneously with this. Squaring the latter,

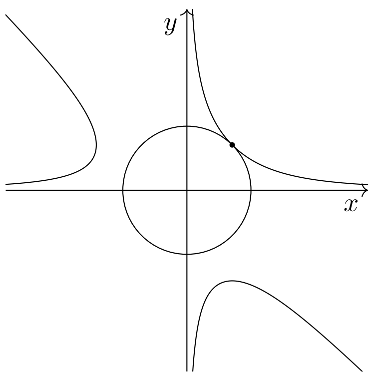
$$\begin{aligned} & x^2y^2\big(x^2 + 2xy + y^2\big) = 256 \\ \Longrightarrow\ & x^2y^2(8 + 2xy) = 256 \\ \Longrightarrow\ & (xy)^3 + 4(xy)^2 - 128 = 0 \\ \Longrightarrow\ & (xy - 4)\big((xy)^2 + 8xy + 32\big) = 0. \end{aligned}$$

The quadratic factor has $\Delta = -64 < 0$, so the only root is $xy = 4$. Substituting this back into $x^2 + y^2 = 8$, we get

$$\begin{aligned} & x^2 + \frac{16}{x^2} = 8 \\ \Longrightarrow\ & \big(x^2 - 4\big)^2 = 0 \\ \Longrightarrow\ & (x - 2)^2(x + 2)^2 = 0. \end{aligned}$$

This gives double roots at $x = \pm 2$. Only the +ve of these lies on $xy(x + y) = 16$, at the point $(2, 2)$. So, the curve is tangential to the circle at a single point, and doesn't cross it elsewhere.

Furthermore, $xy(x + y) = 16$ has asymptotes at $x = 0$, $y = 0$ and $(x + y) = 0$, so each of its three branches is unbounded. Hence, it must always lie on or outside the circle $x^2 + y^2 = 8$. Therefore, if $xy(x + y) = 16$, then $x^2 + y^2 \geq 8$.



4939. Differentiating,

$$\frac{dx}{d\theta} = -4\sin\theta - 4\sin 2\theta.$$

The integral between $\theta = 0$ and $\theta = 2\pi$ is

$$\int_0^{2\pi} (4\sin\theta + 2\sin 2\theta)(-4\sin\theta - 4\sin 2\theta)\, d\theta.$$

The sense of rotation is anticlockwise, which puts positive y with negative $\frac{dx}{d\theta}$. So, the required area is the negative of the above:

$$A = \int_0^{2\pi} (4\sin\theta + 2\sin 2\theta)(4\sin\theta + 4\sin 2\theta)\, d\theta.$$

The integrand is

$$16\sin^2\theta + 12\sin\theta\sin 2\theta + 8\sin^2 2\theta.$$

Using double-angle formulae, this is

$$\begin{aligned} & 8 - 8\cos 2\theta + 24\sin^2\theta\cos\theta + 4 - 4\cos 4\theta \\ \equiv\ & 12 - \underbrace{8\cos 2\theta + 24\sin^2\theta\cos\theta - 4\cos 4\theta}_{*}. \end{aligned}$$

We can now integrate (inspection for the middle term). In the resulting indefinite integral, all the terms marked $*$ are composed of trig functions. Their values are the same at $\theta = 0$ and $\theta = 2\pi$; so, they don't contribute to the definite integral. Only the first term contributes, giving

$$\begin{aligned} A &= \Big[12\theta - ...\Big]_0^{2\pi} \\ &= 24\pi. \end{aligned}$$

4940. Let $z = x^2$. This gives a quintic:

$$z^5 - 2z^4 + z^3 - z^2 + 2z - 2 = 0.$$

Setting the derivative to zero,

$$5z^4 - 8z^3 + 3z^2 - 2z + 2 = 0.$$

This is a quartic. Setting its derivative to zero,

$$\begin{aligned} & 20z^3 - 24z^2 + 6z - 2 = 0 \\ \Longrightarrow\ & 2(z - 1)(10z^2 - 2z + 1) = 0. \end{aligned}$$

The quadratic factor has discriminant -36. So, the cubic has exactly one root, at $z = 1$. The quartic, therefore, has exactly one SP, at $z = 1$. Its value at $z = 1$ is

$$5z^4 - 8z^3 + 3z^2 - 2z + 2\,\Big|_{t=1} = 0.$$

So, the quartic has exactly one root, at $z = 1$. The quintic, therefore, has exactly one SP. This cannot be a turning point, as a quintic must turn twice if it turns at all. So, the SP of the quintic is a point of inflection, meaning that the (positive) quintic is never decreasing. So, it must have exactly one root.

The quintic has value -2 at $z = 0$. So, since it is a positive quintic, its root must satisfy $z = \alpha > 0$. Hence, as required, the order-10 polynomial has exactly two real roots: $x = \pm\sqrt{\alpha}$.

4941. For intersections,

$$x^3 - x^4 = mx + c.$$

For two distinct points of tangency, this equation must have two double roots. So, we can write

$$x^4 - x^3 + mx + c \equiv (x-a)^2(x-b)^2.$$

Equating coefficients,

$$\begin{aligned} x^3 &: -1 = -2a - 2b, \\ x^2 &: 0 = a^2 + 4ab + b^2, \\ x^0 &: c = a^2b^2. \end{aligned}$$

The x^3 equation is $a + b = \frac{1}{2}$. The x^2 equation is $0 = (a+b)^2 + 2ab$, which gives $ab = \frac{1}{8}$. So,

$$\begin{aligned} c &= (ab)^2 \\ &= \tfrac{1}{64}. \end{aligned}$$

4942. (a) The range of the sine function is $[-1, 1]$, which is bounded. So, L cannot be $\pm\infty$.

(b) Consider the following equation:

$$\begin{aligned} &\sin(\ln x) = 1 \\ \implies &\ln x = \tfrac{\pi}{2} + 2n\pi, \text{ for } n \in \mathbb{Z}, \\ \implies &x = e^{\frac{\pi}{2} + 2n\pi}, \text{ for } n \in \mathbb{Z}. \end{aligned}$$

Consider a sub-list of the above roots, with $n = -1, -2, -3, \ldots$. This can be expressed as

$$x = \frac{e^{\frac{\pi}{2}}}{e^{2k\pi}}, \text{ for } k \in \mathbb{N}.$$

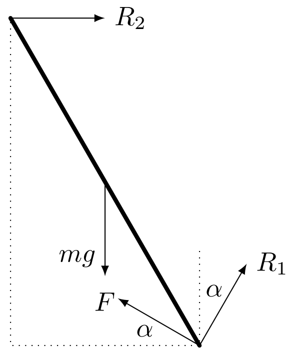
As $k \to \infty$, these x values tend to 0 from above. In other words, there are values of x for which $\sin(\ln x) = 1$ as close to 0 as we choose to find them. Hence, as x tends to zero from above, x keeps passing through values for which $\sin(\ln x) = 1$. Hence, L cannot be zero.

(c) The argument from (b) can be applied, *mutatis mutandis*, to any potential limit $k \in [-1, 1]$. So, the limit is not infinite, nor is it in $[-1, 1]$, nor can it be outside $[-1, 1]$, for the reasons in part (a). So, the limit cannot be defined.

— Nota Bene —

The reason that the limit is undefined is that, as $x \to 0^+$, the value of $\ln x$ tends towards negative infinity. These values form the inputs of the sine function. So, the value of the sine function cycles between -1 and 1 infinitely many times as x gets close to zero. This means, in effect, that the limit approaches all values $k \in [-1, 1]$ at the same time. This can be seen on a graphing calculator: plot the equation $y = \sin(\ln x)$ and zoom into the y axis.

4943. The radius and the length of the ladder are both 1. So, the angle between the radius and the vertical is α. Hence, the angle between the reaction force at the base and the vertical is α, as is the angle between the friction and the horizontal. The force diagram for the ladder is as follows:



Assuming limiting friction, $F = \mu R_1$. Vertically,

$$\begin{aligned} &R_1 \cos\alpha + \mu R_1 \sin\alpha - mg = 0 \\ \implies &R_1 = \frac{mg}{\cos\alpha + \mu \sin\alpha}. \end{aligned}$$

Horizontally,

$$R_1 \sin\alpha - \mu R_1 \cos\alpha + R_2 = 0.$$

Taking moments around the base,

$$\begin{aligned} &2R_2 \cos\alpha - mg \sin\alpha = 0 \\ \implies &R_2 = \tfrac{1}{2} mg \tan\alpha. \end{aligned}$$

Substituting this into the horizontal,

$$\begin{aligned} &R_1 \sin\alpha - \mu R_1 \cos\alpha + \tfrac{1}{2} mg \tan\alpha = 0 \\ \implies &R_1 = \frac{mg \tan\alpha}{2(\mu\cos\alpha - \sin\alpha)}. \end{aligned}$$

Equating the expressions for R_1,

$$\begin{aligned} &\frac{mg}{\cos\alpha + \mu\sin\alpha} = \frac{mg\tan\alpha}{2(\mu\cos\alpha - \sin\alpha)} \\ \implies &2(\mu\cos\alpha - \sin\alpha) = \sin\alpha + \mu\sin\alpha\tan\alpha \\ \implies &\mu(2\cos\alpha - \sin\alpha\tan\alpha) = 3\sin\alpha \\ \implies &\mu = \frac{3\sin\alpha}{2\cos\alpha - \sin\alpha\tan\alpha}. \end{aligned}$$

This is for limiting friction. For equilibrium, the coefficient of friction must exceed the above value. Multiplying top and bottom by $2\cos\alpha$,

$$\begin{aligned} \mu &\geq \frac{6\sin\alpha\cos\alpha}{4\cos^2\alpha - 2\sin^2\alpha} \\ &\equiv \frac{3\sin 2\alpha}{3\cos^2\alpha - 3\sin^2\alpha + \cos^2\alpha + \sin^2\alpha} \\ &\equiv \frac{3\sin 2\alpha}{3\cos 2\alpha + 1}, \text{ as required.} \end{aligned}$$

4944. Using the Euler substitution, let

$$\sqrt{x^2+k} = -x + t.$$

Differentiating this with respect to x,

$$\begin{aligned} x(x^2+k)^{-\frac{1}{2}} &= -1 + \frac{dt}{dx} \\ \implies \frac{dt}{dx} &= \frac{x}{\sqrt{x^2+k}} + 1 \\ &\equiv \frac{x+\sqrt{x^2+k}}{\sqrt{x^2+k}} \\ &\equiv \frac{t}{\sqrt{x^2+k}} \\ \implies \frac{1}{t}\,dt &= \frac{1}{\sqrt{x^2+k}}\,dx. \end{aligned}$$

Enacting the substitution,

$$\begin{aligned} &\int \frac{1}{\sqrt{x^2+k}}\,dx \\ &= \int \frac{1}{t}\,dt \\ &= \ln|t| + c \\ &= \ln\left|x+\sqrt{x^2+k}\right| + c, \text{ as required.} \end{aligned}$$

4945. We prove this by construction. The solution set of $\sin\theta > 1/2$ includes the primary interval $(\pi/6, 5\pi/6)$. To 3sf, this is $(0.534, 2.62)$, so it contains $[1, 2]$. Hence, set R defined as follows is a subset of S:

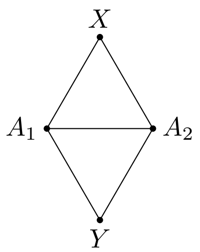
$$R = \left\{x \in \mathbb{Q} : 1 \le x^2 \le 2\right\}.$$

The following are all elements of R:

$$\{1, 1.1, 1.01, 1.001, 1.0001, ...\}.$$

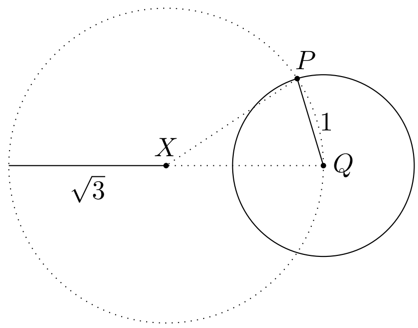
Each of these elements is also in S. So, the set S has infinitely many elements. □

4946. Assume, for a contradiction, that every pair of points 1 unit apart are coloured differently. Wlog, colour a point X red, and construct two equilateral triangles of side length 1 as shown.



Neither A_1 nor A_2 is red. And, since they are a distance of 1 unit apart, they cannot be the same colour. Hence, one must be blue and the other green. In turn, this means that point Y is red. The distance between X and Y is $\sqrt{3}$.

Construct a circle, centred on X, of radius $\sqrt{3}$. Then construct a circle of radius 1, centred on a point on the circumference.



Since points P and Q lie on the dotted circle, they are both $\sqrt{3}$ from X, and are therefore both red. And they are 1 unit apart. This is a contradiction. Hence, there must be points of the same colour separated by 1 unit. □

4947. (a) True.

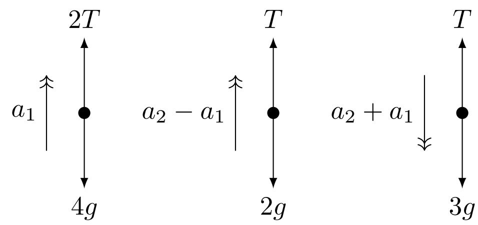
(b) False: $x^{10}+1=0$ is a counterexample.

(c) True: the quintic must have a root $x^3 = k$. This is a cubic equation, which must itself have a real root.

4948. (a) i. Smooth pulleys and light strings.

ii. The movable pulley is light. This means that it cannot have any resultant force on it, so its equation of motion is $2T_1 - T_2 = 0$. The tension in the upper string is twice the tension in the lower string.

(b) Call the rightwards acceleration of the upper string over its pulley a_1, and likewise with a_2 for the lower. The force diagrams for the three masses are:



The equations of motion are

$$\begin{aligned} 2T - 4g &= 4a_1, \\ T - 2g &= 2(a_2 - a_1), \\ 3g - T &= 3(a_2 + a_1). \end{aligned}$$

Subtracting twice the second from the first,

$$\begin{aligned} 0 &= 8a_1 - 4a_2 \\ \implies a_2 &= 2a_1. \end{aligned}$$

Adding the last two equations,

$$g = 5a_2 + a_1.$$

Substituting for a_2,

$$\begin{aligned} g &= 10a_1 + a_1 \\ \implies a_1 &= \tfrac{1}{11}g. \end{aligned}$$

So, the 4 kg mass accelerates at $\frac{1}{11}g$ ms^{-2}.

4949. On a Venn diagram of the possibility space, there are ${}^4\text{C}_2 = 6$ regions in which two events occur, ${}^4\text{C}_1 = 4$ regions in which one event occurs, and ${}^4\text{C}_0 = 1$ region in which no events occur. So, there are 11 relevant probabilities.

(a) Assume, for a contradiction, that

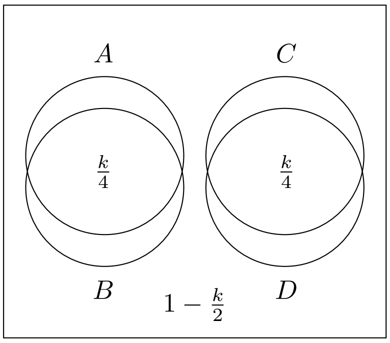
$$\mathbb{P}(A \cap B) > \tfrac{1}{2}.$$

The sum of the other 10 probabilities must be less than $1/2$, meaning that e.g. $\mathbb{P}(C) < 1/2$. Since $\mathbb{P}(A) = \mathbb{P}(C)$, this is a contradiction. So, $\mathbb{P}(A \cap B) \leq 1/2$.

Consider the case in which all 11 probabilities are zero, except

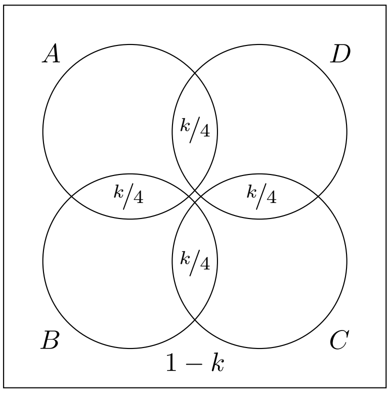
$$\begin{aligned}\mathbb{P}(A \cap B) = \mathbb{P}(C \cap D) = \tfrac{k}{4},\\ \mathbb{P}(A' \cap B' \cap C' \cap D') = 1 - \tfrac{k}{2}.\end{aligned}$$

The Venn diagram is



With $k \in [0, 1]$, this satisfies the conditions of the problem. Hence, the set of possible values for $\mathbb{P}(A \cap B)$ is $[0, 1/2]$.

(b) Set four of the two-way intersections, namely $\mathbb{P}(A \cap B)$, $\mathbb{P}(B \cap C)$, $\mathbb{P}(C \cap D)$ and $\mathbb{P}(D \cap A)$ to $\frac{k}{4}$, and set $\mathbb{P}(A' \cap B' \cap C' \cap D')$ to $1 - k$. The Venn diagram is



With $k \in [0, 1]$, this satisfies the conditions of the problem, and gives $\mathbb{P}(A \cup B \cup C) = k$. So, the set of possible values for $\mathbb{P}(A \cup B \cup C)$ is $[0, 1]$.

4950. The domain of definition is $[-1, 1]$. Both sides of the identity have odd symmetry, so we need only prove the result for $x \in [0, 1]$.

$$\begin{aligned}&\sin\theta \equiv 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\\ \Longrightarrow\ &\sin^2\theta \equiv 4\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta.\end{aligned}$$

Let $s = \sin\frac{1}{2}\theta$:

$$\begin{aligned}&\sin^2\theta \equiv 4s^2(1 - s^2)\\ \Longrightarrow\ &s^4 - s^2 + \tfrac{1}{4}\sin^2\theta = 0\\ \Longrightarrow\ &s^2 = \frac{1 \pm \sqrt{1 - \sin^2\theta}}{2}.\end{aligned}$$

Given $\theta = 0, s = 0$, we want the negative root:

$$\sin^2\tfrac{1}{2}\theta \equiv \frac{1 - \sqrt{1 - \sin^2\theta}}{2}.$$

Let $\theta = \arcsin x$:

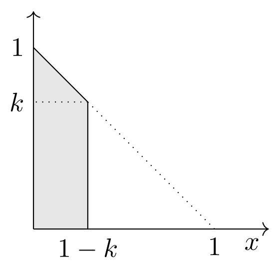
$$\sin^2\left(\tfrac{1}{2}\arcsin x\right) = \frac{1 - \sqrt{1 - x^2}}{2}.$$

The square of the original RHS is

$$\begin{aligned}&\left(\tfrac{1}{2}\left(\sqrt{1 + x} - \sqrt{1 - x}\right)\right)^2\\ \equiv\ &\tfrac{1}{4}\left(1 + x + (1 - x) - 2\sqrt{1 - x^2}\right)\\ \equiv\ &\tfrac{1}{4}\left(2 - 2\sqrt{1 - x^2}\right)\\ \equiv\ &\frac{1 - \sqrt{1 - x^2}}{2}.\end{aligned}$$

Over the domain $[0, 1]$, we can take the positive square root. Combined with the aforementioned symmetry, this proves the result for $x \in [-1, 1]$.

4951. At height $z = k$, the (x, y) cross-section is



The area of the trapezium is

$$A_k = \tfrac{1}{2}(1 - k)(1 + k) \equiv \tfrac{1}{2}(1 - k^2).$$

To find the volume of the sculpture, we integrate this between $k = 0$ and $k = 1$. This gives

$$\begin{aligned}V_{\text{sculpture}} &= \int_{k=0}^{k=1} \tfrac{1}{2}(1 - k^2)\,dk\\ &= \left[\tfrac{1}{2}k - \tfrac{1}{6}k^3\right]_{k=0}^{k=1}\\ &= \tfrac{1}{3}.\end{aligned}$$

4952. Projected at speed u and angle θ, the equation for the time to maximum height is $0 = u\sin\theta - gt$. So, the time at which sparkling occurs is

$$t = \frac{u\sin\theta}{g}.$$

The coordinates at this point are

$$\left(\frac{u^2\sin 2\theta}{g}, \frac{u^2\sin^2\theta}{2g}\right).$$

The latter can be written as

$$y = \frac{u^2(1-\cos 2\theta)}{4g}.$$

Rearranging the coordinate equations,

$$\begin{aligned}\sin 2\theta &= \frac{gx}{u^2},\\ \cos 2\theta &= 1 - \frac{4gy}{u^2}.\end{aligned}$$

Squaring and adding these,

$$\frac{g^2x^2}{u^4} + \left(1 - \frac{4gy}{u^2}\right)^2 = 1.$$

This is the equation of an ellipse, as required.

4953. There are n^2 grid squares, giving ${}^{n^2}\mathrm{C}_{n-2}$ as the number of outcomes in the possibility space. We need to show that there are $n(n^2+3)$ outcomes in which the counters are collinear. We classify these by the total length of the line involved, including any blank grid squares outside the counters.

① Of length n, there are n rows, n columns and two diagonals. In each, there are ${}^{n}\mathrm{C}_{n-2}$ ways of positioning the counters. The number of outcomes is

$$\begin{aligned}&(2n+2)\times \tfrac{1}{2}n(n-1)\\ \equiv\,& n(n-1)(n+1).\end{aligned}$$

② Of length $n-1$, there are four diagonals. In each, there are ${}^{n-1}\mathrm{C}_{n-2}$ ways of positioning the counters. This gives $4(n-1)$ outcomes.

③ Of length $n-2$, there are four diagonals. Each gives one outcome.

So, the total number of collinear outcomes is

$$\begin{aligned}&n(n+1)(n-1) + 4(n-1) + 4\\ \equiv\ & n(n^2+3).\end{aligned}$$

Dividing by the total number of outcomes, we reach the required result:

$$p = \frac{n(n^2+3)}{{}^{n^2}\mathrm{C}_{n-2}}.$$

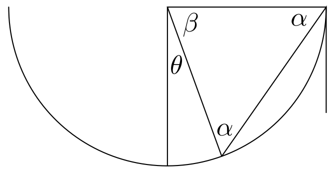
4954. We define new coordinate variables as follows:

$$\begin{aligned}p &= \tfrac{1}{\sqrt{2}}(y+x),\\ q &= \tfrac{1}{\sqrt{2}}(y-x).\end{aligned}$$

The (p,q) axes are at $45°$ to the (x,y) axes, and have the same scaling. In terms of p and q, the transformed graph is $q = \mathrm{f}(p)$. In terms of x and y, it is

$$\tfrac{1}{\sqrt{2}}(y-x) = \mathrm{f}\left(\tfrac{1}{\sqrt{2}}(y-x)\right).$$

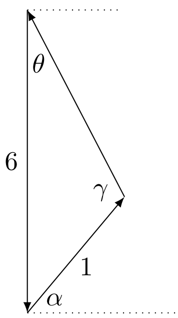
4955. (a) The circle geometry is



Angle β is $90° - \theta$. So,

$$\begin{aligned}\alpha &= 90° - \tfrac{1}{2}\beta\\ &= 90° - \tfrac{1}{2}(90° - \theta)\\ &\equiv \tfrac{1}{2}\theta + 45°.\end{aligned}$$

The triangle of forces for A, scaled down by mg, is as follows:



Using alternate angles,

$$\begin{aligned}\gamma &= 90° - \theta + \alpha\\ &= 135° - \tfrac{1}{2}\theta.\end{aligned}$$

Using the sine rule,

$$\begin{aligned}&6\sin\theta = \sin(135° - \tfrac{1}{2}\theta)\\ \Longrightarrow\ &6\sin\theta = \tfrac{\sqrt{2}}{2}\cos\tfrac{1}{2}\theta + \tfrac{\sqrt{2}}{2}\sin\tfrac{1}{2}\theta\\ \Longrightarrow\ &6\sqrt{2}\sin\theta = \cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta.\end{aligned}$$

(b) Squaring both sides,

$$72\sin^2\theta = \cos^2\tfrac{1}{2}\theta + 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta.$$

Simplifying, we get a quadratic in $\sin\theta$:

$$\begin{aligned}&72\sin^2\theta = 1 + \sin\theta\\ \Longrightarrow\ &72\sin^2\theta - \sin\theta - 1 = 0\\ \Longrightarrow\ &(8\sin\theta - 1)(9\sin\theta + 1) = 0\\ \Longrightarrow\ &\sin\theta = \tfrac{1}{8}, -\tfrac{1}{9}.\end{aligned}$$

Since $\theta \in (0, \pi/2)$, we reject the negative root, which gives $\theta = \arcsin\frac{1}{8}$, as required.

4956. (a) Start with $\sec x + \tan x \geq 0$:

$$\begin{aligned} &\frac{d}{dx}\ln(\sec x + \tan x) \\ \equiv\; &\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ \equiv\; &\frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} \\ \equiv\; &\sec x. \end{aligned}$$

Then with $\sec x + \tan x < 0$:

$$\begin{aligned} &\frac{d}{dx}\ln(-\sec x - \tan x) \\ \equiv\; &\frac{-\sec x \tan x - \sec^2 x}{-\sec x - \tan x} \\ \equiv\; &\sec x. \end{aligned}$$

Combining both results,

$$\int \sec x = \ln|\sec x + \tan x| + c, \text{ as required.}$$

(b) Let $u = \sec x$ and $v' = \sec^2 x$. From these, we get $u' = \sec x \tan x$ and $v = \tan x$. Quoting the parts formula,

$$\begin{aligned} I &= \int \sec^3 x\, dx \\ &= \sec x \tan x - \int \sec x \tan^2 x\, dx \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx \\ &= \sec x \tan x - \int \sec^3\, dx + \int \sec x\, dx. \end{aligned}$$

The integral in the middle is $-I$. So,

$$2I = \sec x \tan x + \int \sec x\, dx.$$

Quoting the result from part (a),

$$I = \tfrac{1}{2}\big(\sec x \tan x + \ln|\sec x + \tan x|\big) + d.$$

4957. Assume that $2^k + 1$ is prime, and that k can be factorised as $k = ab$ for $a, b > 1$. This gives

$$2^k + 1 = 2^{ab} + 1.$$

If a is odd, then we can factorise as follows:

$$\begin{aligned} &2^{ab} + 1 \\ =\; &\big(2^b + 1\big)\big(\underbrace{2^{(a-1)b} - 2^{(a-2)b} + ... - 2^b + 1}_{a \text{ terms}}\big) \end{aligned}$$

But $2^{ab} + 1$ is prime, so a must be even. And a and b appear symmetrically, so the same argument applies to b. Hence, any factors of k are even. This implies that k is a power of 2. □

4958. Reflection in $z = 1$ is equivalent to reflection in $z = 0$ and then translation by $2\mathbf{k}$. This gives

$$z = -\sin(x + y) - \sin(x - y) + 2.$$

Translating by vector $\pi\mathbf{i} + 2\pi\mathbf{j} - 2\mathbf{k}$, we replace x by $x - \pi$, y by $y - 2\pi$ and z by $z + 2$. This gives

$$\begin{aligned} z + 2 &= -\sin(x - \pi + y - 2\pi) \\ &\quad - \sin(x - \pi - (y - 2\pi) + 2 \\ \Longrightarrow z &= -\sin(x + y - 3\pi) - \sin(x - y + \pi). \end{aligned}$$

We now use the following identities:

$$\begin{aligned} \sin(\theta + 2\pi) &\equiv \sin\theta, \\ \sin(\theta + \pi) &\equiv -\sin\theta. \end{aligned}$$

These give the new equation as

$$\begin{aligned} z &= -\sin(x + y + \pi) - \sin(x - y + \pi) \\ &\equiv \sin(x + y) + \sin(x - y). \end{aligned}$$

This is the same as the original surface.

4959. (a) i. Using the variable p to enact the integral, keeping x for its limits, the cross-sectional area is given by

$$\begin{aligned} A &= \int_{-x}^{x} \tfrac{3}{4}x^2 - \tfrac{3}{4}p^2\, dp \\ &\equiv \Big[\tfrac{3}{4}x^2 p - \tfrac{1}{4}p^3\Big]_{-x}^{x} \\ &\equiv 2\big(\tfrac{3}{4}x^3 - \tfrac{1}{4}x^3\big) \\ &\equiv x^3. \end{aligned}$$

ii. Call the length of the ditch l. The rate of gain of cross-sectional area due to influx is 3 m^3 per metre per month, which is $3l$ m^3 per month. The surface of the water is then a rectangle measuring $2x$ by l, with area $S = 2xl$. So, the loss per month is $1.5 \times 2xl = 3xl$. These combine to give

$$\begin{aligned} \frac{dV}{dt} &= 3l - 3xl \\ &\equiv 3l(1 - x). \end{aligned}$$

In terms of cross-sectional area, the volume is $V = Al$. Differentiating this,

$$\frac{dV}{dt} = \frac{dA}{dt}l.$$

Putting the results together,

$$\begin{aligned} &\frac{dA}{dt}l = 3l(1 - x) \\ \Longrightarrow\; &\frac{dA}{dt} = 3(1 - x). \end{aligned}$$

(b) Substituting $A = x^3$ in,

$$\begin{aligned} &\frac{d}{dt}(x^3) = 3(1 - x) \\ \Longrightarrow\; &3x^2\frac{dx}{dt} = 3(1 - x) \\ \Longrightarrow\; &\int \frac{x^2}{1 - x}\, dx = \int 1\, dt. \end{aligned}$$

We rewrite the x integrand $\dfrac{x^2}{1-x}$ as

$$\begin{aligned}&\frac{-x(1-x)-1(1-x)+1}{1-x}\\ \equiv\ &-x-1+\frac{1}{1-x}.\end{aligned}$$

This gives

$$\begin{aligned}-\tfrac{1}{2}x^2-x-\ln|1-x|&=t+c\\ \therefore\quad \tfrac{1}{2}x^2+x+\ln|1-x|&=d-t.\end{aligned}$$

For large t, $d-t\to-\infty$. Since the quadratic $\frac{1}{2}x^2+x$ is bounded below, we need

$$\begin{aligned}\ln|1-x|&\approx d-t\\ \Longrightarrow |1-x|&\approx e^{d-t}.\end{aligned}$$

Whatever the value of d, x must tend to 1 for large t. This gives a depth y of 0.75 m or 75 cm, as required.

4960. Since $\mathrm{f}_1(x)$ and $\mathrm{f}_2(x)$ are both solutions of the DE, their second derivatives $\mathrm{f}_1''(x)$ and $\mathrm{f}_2''(x)$ are both identically equal to $\mathrm{g}(x)$. So, they are identically equal to each other:

$$\mathrm{f}_1''(x)-\mathrm{f}_2''(x)\equiv 0.$$

Integrating this twice, the following holds for some constants c and d:

$$\mathrm{f}_1(x)-\mathrm{f}_2(x)\equiv cx+d.$$

The intersections of $y=\mathrm{f}_1(x)$ and $y=\mathrm{f}_2(x)$ are roots of the equation

$$\mathrm{f}_1(x)-\mathrm{f}_2(x)=0.$$

This is $cx+d=0$, which is a linear equation. It cannot have infinitely many roots, because f_1 and f_2 are distinct. So, it has a maximum of one root. This gives a maximum of one intersection between $y=\mathrm{f}_1(x)$ and $y=\mathrm{f}_2(x)$.

So, she would see the same effect.

4961. Place A wlog. Then consider the outcomes as a list of the 5! orders of the remaining five letters. Success requires exactly two of D,E and F to be next to one another. So, failure requires either all three together, or all three separated:

① ALL TOGETHER. There are $3!=6$ orders of $\{DEF\},B,C$, and then 3! orders of D,E,F within that, giving 36 outcomes.

② ALL SEPARATED. There is one set of locations for D,E,F, with 3! orders within it, and one pair of locations for B,C, with two orders within it. This gives $3!\times 2!=12$ outcomes.

So, the probability of failure is $\frac{48}{5!}=\frac{2}{5}$, and the probability of success is $\frac{3}{5}$.

——— ALTERNATIVE METHOD ———

The exact positions of the points isn't relevant, only their order around the circumference is. So, without loss of generality, we can place A,B,C in a triangle, and D between A and B. Consider the location of E. With probability $\frac{1}{2}$, E is next to D.

① With E next to D, success requires F to be apart from both of them. There are 5 regions to choose from, of which 2 are successful.

② With E apart from D, success requires F to be next to either E or D. There are 5 regions to choose from, of which 4 are successful.

So, the overall probability is

$$\begin{aligned}p&=\tfrac{1}{2}\times\tfrac{2}{5}+\tfrac{1}{2}\times\tfrac{4}{5}\\ &=\tfrac{3}{5}.\end{aligned}$$

4962. The shortest distance must lie along a normal to both surfaces. This is a normal to the plane which passes through the origin, i.e. the line

$$x=y=z.$$

The closest point to the origin is (a,a,a), where $a+a+a=9$. This is $(3,3,3)$, which is a distance $\sqrt{27}$ from O. The radius of the sphere is 3, so the shortest distance is $\sqrt{27}-3$.

4963. The golden ratio ϕ is a root of the equation

$$x^2-x-1=0.$$

So, we know that

$$\phi^2=\phi+1.$$

Dividing through by ϕ, it also tells us that

$$\begin{aligned}\phi&=1+\phi^{-1}\\ \Longrightarrow 1-\phi&=\phi^{-1}.\end{aligned}$$

Using these results, the RHS is

$$\begin{aligned}&F_n+F_{n+1}\\ =\ &\frac{\phi^n-(-\phi)^{-n}+\phi^{n+1}-(-\phi)^{-n-1}}{\sqrt{5}}\\ \equiv\ &\frac{\phi^n(1+\phi)-(-\phi)^{-n-1}(-\phi+1)}{\sqrt{5}}\\ \equiv\ &\frac{\phi^n\phi^2-(-\phi)^{-n-1}\phi^{-1}}{\sqrt{5}}\\ \equiv\ &\frac{\phi^{n+2}-(-\phi)^{-n-2}}{\sqrt{5}}\\ =\ &F_{n+2}.\end{aligned}$$

Since it also produces the correct starting values $F_1=F_2=1$, this ordinal definition generates the Fibonacci sequence. □

4964. The RHS is the number of ways of choosing $(r+1)$ people from a group of $(n+1)$.

Label the people $1, ..., n+1$. We classify the ways of picking the committee by the least label picked. If label 1 is picked out, then there are ${}^n\text{C}_r$ ways of picking the remaining r. If label 2 is the least label picked, then there are ${}^{n-1}\text{C}_r$ ways. Continuing in this vein, the full classification is

Smallest picked	Number of ways
1	${}^n\text{C}_r$
2	${}^{n-1}\text{C}_r$
3	${}^{n-2}\text{C}_r$
...	...
$n-r+1$	${}^r\text{C}_r$.

Adding these up (from the bottom of the table to the top), we get

$$\sum_{i=r}^{n} {}^i\text{C}_r.$$

Equating the two results for the number of ways of picking the committee proves the result:

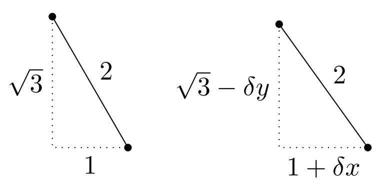
$$\sum_{i=r}^{n} {}^i\text{C}_r \equiv {}^{n+1}\text{C}_{r+1}.$$

— Nota Bene —

This is called the *hockey-stick identity*; locate the sum and summands on Pascal's triangle to see why.

4965. Wlog, let $r=1$. Call the acceleration of the upper core a ms^{-2}. Initially, a line of centres between the upper and lower cores has angle of inclination 60°, with components of length $\sqrt{3}$ vertically and 1 horizontally.

Suppose the upper core moves downwards a small distance δy and a lower core moves sideways by δx. The distance between the centres is still 2.



Using Pythagoras,

$$\begin{aligned} &(1+\delta x)^2 + (\sqrt{3}-\delta y)^2 = 2^2 \\ \Longrightarrow\ &1 + 2\delta x + \delta x^2 + 3 - 2\sqrt{3}\delta y + \delta y^2 = 4 \\ \Longrightarrow\ &2\delta x + \delta x^2 - 2\sqrt{3}\delta y + \delta y^2 = 0. \end{aligned}$$

The changes are small, so the quadratic terms can be neglected. This gives

$$\begin{aligned} &2\delta x - 2\sqrt{3}\delta y \approx 0 \\ \Longrightarrow\ &\delta x \approx \sqrt{3}\delta y. \end{aligned}$$

Hence, at the instant the rope breaks, the ratio of accelerations is $\sqrt{3}:1$.

Resolving vertically for the upper core,

$$\begin{aligned} &mg - 2R\sin 60° = ma \\ \Longrightarrow\ &mg - \sqrt{3}R = ma. \end{aligned}$$

Resolving horizontally for a lower core,

$$\begin{aligned} &R\sin 30° = \sqrt{3}ma \\ \Longrightarrow\ &R = 2\sqrt{3}ma. \end{aligned}$$

Substituting into the upper-core equation,

$$\begin{aligned} &mg - \sqrt{3}\cdot 2\sqrt{3}ma = ma \\ \Longrightarrow\ &a = \tfrac{1}{7}g, \text{ as required.} \end{aligned}$$

4966. Let $t = \tan\frac{x}{2}$. Writing $\sin x$ in terms of t,

$$\begin{aligned} \sin x &\equiv 2\sin\tfrac{x}{2}\cos\tfrac{x}{2} \\ &\equiv \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} \\ &\equiv \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \\ &= \frac{2t}{1+t^2}. \end{aligned}$$

And writing $\cos x$ in terms of t,

$$\begin{aligned} \cos x &\equiv \cos^2\tfrac{x}{2} - \sin^2\tfrac{x}{2} \\ &\equiv \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} \\ &\equiv \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \\ &= \frac{1-t^2}{1+t^2}. \end{aligned}$$

For the change of variable of integration,

$$\begin{aligned} \frac{dt}{dx} &= \tfrac{1}{2}\sec^2\tfrac{x}{2} \\ &\equiv \tfrac{1}{2}\left(1+\tan^2\tfrac{x}{2}\right) \\ &= \tfrac{1}{2}\left(1+t^2\right). \end{aligned}$$

This gives

$$dx = \frac{2\,dt}{1+t^2}.$$

Enacting the Weierstrass substitution,

$$\int \text{f}\,(\sin x, \cos x)\ dx$$

can be written

$$\int \text{f}\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2\,dt}{1+t^2}, \text{ as required.}$$

4967. (a) The derivatives of the quadratic are $2ax + b$ and $2a$. So, in order to match the value of the function and its first two derivatives at $x = k$, we require

$$\begin{aligned}\mathrm{f}''(k) &= 2a,\\ \mathrm{f}'(k) &= 2ak + b,\\ \mathrm{f}(k) &= ak^2 + bk + c.\end{aligned}$$

From the top, this gives

$$\begin{aligned}a &= \tfrac{1}{2}\,\mathrm{f}''(k),\\ b &= \mathrm{f}'(k) - k\,\mathrm{f}''(k),\\ c &= \mathrm{f}(k) - k(\mathrm{f}'(k) - k\,\mathrm{f}''(k)) - \tfrac{1}{2}k^2\,\mathrm{f}''(k)\\ &\equiv \mathrm{f}(k) - k\,\mathrm{f}'(k) + \tfrac{1}{2}k^2\,\mathrm{f}''(k).\end{aligned}$$

(b) For $\mathrm{f}(x) = \ln x + x$, the derivatives are

$$\begin{aligned}\mathrm{f}'(x) &= \tfrac{1}{x} + 1,\\ \mathrm{f}''(x) &= -\tfrac{1}{x^2}.\end{aligned}$$

Substituting $x = k$, this gives

i. $a = -\frac{1}{2k^2}$.

ii. $b = \frac{1}{k} + 1 - k \cdot -\frac{1}{k^2}$.
$\equiv \frac{2}{k} + 1$

iii. $c = \ln k + k - k\left(\frac{1}{k} + 1\right) + \frac{1}{2}k^2 \cdot -\frac{1}{k^2}$
$= \ln k - \frac{3}{2}$

(c) At $k = 0.5$, the coefficients are

$$\begin{aligned}a &= -\tfrac{1}{2\times 0.5^2} = -2,\\ b &= \tfrac{2}{0.5} + 1 = 5,\\ c &= \ln\tfrac{1}{2} - \tfrac{3}{2} = -\ln 2 - \tfrac{3}{2}.\end{aligned}$$

So, the approximating parabola is

$$y = -2x^2 + 5x - \ln 2 - \tfrac{3}{2}.$$

Setting $y = 0$ to find the new approximation,

$$\begin{aligned}&-2x^2 + 5x - \ln 2 - \tfrac{3}{2} = 0\\ \Longrightarrow\ & 2x^2 - 5x + \ln 2 + \tfrac{3}{2} = 0\\ \Longrightarrow\ & x = \frac{5 \pm \sqrt{25 - 8(\ln 2 + 3/2)}}{4}\\ & = \frac{5 \pm \sqrt{13 - 8\ln 2}}{4}.\end{aligned}$$

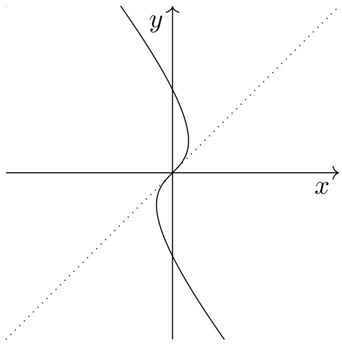
Taking the $-$ve root, the new approximation is at

$$x_1 = \frac{5 - \sqrt{13 - 8\ln 2}}{4} \approx 0.567412.$$

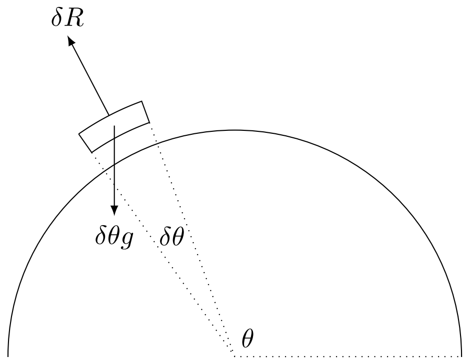
4968. Adding the equations, $x + y = 2t$. Subtracting them, $x - y = 2t^3$. Let $X = x + y$ and $Y = x - y$. This gives parametric equations, in the (X, Y) plane, $X = 2t$ and $Y = 2t^3$, which has Cartesian equation $Y = \frac{1}{4}X^3$. The second derivative is

$$\frac{d^2Y}{dX^2} = \tfrac{3}{2}X.$$

This changes sign at $X = 0$, which is the origin.



4969. In NII along the rope, the tensions, which are non-constant but internal, must cancel by NIII. We only need consider the component of the weight acting along the tangent. Consider a small section of rope subtending angle $\delta\theta$ at the centre. Wlog, let its mass be $\delta\theta$.



In the limit of small $\delta\theta$, the tangential component of the weight of this piece of rope is

$$\delta\theta g \sin\left(\theta - \tfrac{\pi}{2}\right) = -g\cos\theta\,\delta\theta.$$

So, the total resultant force is given by

$$F = \lim_{\delta\theta\to 0}\sum_{\theta=\frac{\pi}{3}}^{\theta=\frac{5\pi}{6}} -g\cos\theta\,\delta\theta.$$

Taking the limit, this becomes an integral:

$$\begin{aligned}F &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} -g\cos\theta\,d\theta\\ &= \Big[-g\sin\theta\Big]_{\frac{\pi}{3}}^{\frac{5\pi}{6}}\\ &= -\tfrac{1}{2}g + \tfrac{\sqrt{3}}{2}g\\ &= \frac{\sqrt{3}-1}{2}g.\end{aligned}$$

The total mass of the rope is given by the total angle subtended at the centre, which is $\frac{1}{2}\pi$. NII is

$$\begin{aligned}&\frac{\sqrt{3}-1}{2}g = \tfrac{1}{2}\pi a\\ \Longrightarrow\ & a = \frac{\sqrt{3}-1}{\pi}g \text{ ms}^{-2}, \text{ as required.}\end{aligned}$$

4970. The cubic and therefore its derivative both have integer coefficients. So, the gradient at $x = a$ is an integer. And the equation of the tangent line is $y = m(x-a)+\mathrm{f}(a)$. When written as $y = mx+c$, both m and c are integers.

Consider the equation for intersections between the curve and the tangent line:

$$\mathrm{f}(x) - \mathrm{g}(x) = 0.$$

This is a monic cubic with integer coefficients. We already know that it has a double root at $x = a$. So, it must have a factor of $(x-a)^2$. Taking this factor out, the equation for intersections is

$$\begin{aligned} &(x-a)^2(x-b) = 0 \\ \Longleftrightarrow\ & x^3 - (2a+b)x^2 + (a^2+2ab)x - a^2b = 0. \end{aligned}$$

We already know that all coefficients are integers. So, $2a+b$ is an integer. And so is $2a$. Hence, b must be an integer. □

4971. When the distance from the origin is greatest or least, the tangent vector is perpendicular to the position vector. In other words,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Differentiating the original curve,

$$2(y-x)\left(\frac{dy}{dx} - 1\right) + 2y\frac{dy}{dx} = 0.$$

Substituting the previous equation in,

$$\begin{aligned} &2(y-x)\left(-\tfrac{x}{y} - 1\right) + 2y \cdot -\tfrac{x}{y} = 0 \\ \Longrightarrow\ & y^2 + xy - x^2 = 0 \\ \Longrightarrow\ & y = \frac{-x \pm \sqrt{x^2+4x^2}}{2} \\ &\equiv \frac{-1 \pm \sqrt{5}}{2}x. \end{aligned}$$

The lines through the points of greatest and least distance from the origin are the lines of symmetry. These are $2y = \big(-1 \pm \sqrt{5}\big)x$, as required.

4972. Assume, for a contradiction, that

① $2^p - 1$ is prime,

② p is not prime, and can be written as $p = ab$, where a and b are integers greater than 1.

This gives

$$\begin{aligned} &2^p - 1 \\ &= 2^{ab} - 1 \\ &= \big(2^a - 1\big)\big(1 + 2^a + 2^{2a} + ... + 2^{a(b-1)}\big). \end{aligned}$$

Since $a, b > 1$, both factors above are greater than 1. So, $2^p - 1$ is not prime. This is a contradiction. Hence, if $2^p - 1$ is prime, then p is prime. QED.

4973. Consider the asymptote at the x axis ($y = 0$) as $x \to \infty$. Assume, for a contradiction, that $a \neq 0$. Whatever the value of the other constants b, c, d, e, the term ax^4 must eventually dominate. And, since it tends to infinity, there is a point beyond which the LHS cannot equal 4. So, the asymptote at the x axis (which continues to produce points all the way to infinity) tells us that $a = 0$, i.e. that the LHS has a factor of y.

Generalising this argument, the asymptote at the y axis ($x = 0$) necessitates a factor of x, and the asymptotes at $x \pm y = 0$ necessitate factors of $(x \pm y)$. So, the curve must be

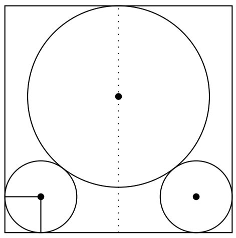
$$xy(x+y)(x-y) = k.$$

Multiplying out and scaling to ensure that the curve is tangent to the circle, the equation of the curve is

$$x^3y - xy^3 = 4.$$

This gives $a = 0$, $b = 1$, $c = 0$, $d = -1$, $e = 0$.

4974. If a circle intersects the line of symmetry, then its centre must lie on it. If a circle doesn't intersect the line of symmetry, then it must be mirrored on the opposite side. It is clear that having all three centres on the line of symmetry is not maximal. So, we centre one circle on the line of symmetry, and put the other two symmetrically either side of it. The optimal arrangement must be of the following type:



Call the radii R, r, r. The vertical distance between the centres is given by

$$\begin{aligned} &\sqrt{(R+r)^2 - \left(\tfrac{1}{2} - r\right)^2} \\ \equiv\ & \tfrac{1}{2}\sqrt{(2R+1)(2R+4r-1)}. \end{aligned}$$

So, equating expressions for the square's height,

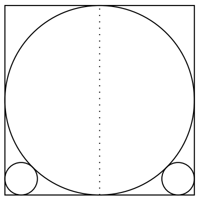
$$\begin{aligned} &R + r + \tfrac{1}{2}\sqrt{(2R+1)(2R+4r-1)} = 1 \\ \Longrightarrow\ & \sqrt{(2R+1)(2R+4r-1)} = 2 - 2R - 2r. \end{aligned}$$

This simplifies to $8R = 4r^2 - 12r + 5$. So, the total area is given by

$$\begin{aligned} &A_{\text{total}} = \pi\big(R^2 + 2r^2\big) \\ \Longrightarrow\ & \tfrac{1}{\pi}A_{\text{total}} = \tfrac{1}{64}(4r^2 - 12r + 5)^2 + 2r^2 \\ \Longrightarrow\ & \tfrac{64}{\pi}A_{\text{total}} = 16r^4 - 96r^3 + 312r^2 - 120r + 25. \end{aligned}$$

Analysing this quartic, we find that it has only one SP, a minimum at $r \approx 0.212$. Hence, the quartic is maximised at its boundaries, which are defined by the fact that $R \le 1/2$ and $r \ge 1/4$. The maximum value for the area can only, therefore, be attained at one of the following boundary cases:

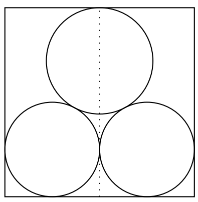
① With R as large as possible:



In this case, $R = 1/2$ and $r = \frac{1}{2}\big(3 - 2\sqrt{2}\big)$. This gives a total area of

$$\begin{aligned} A_{\text{total}} &= \tfrac{1}{4}\pi\left(1 + 2\left(3 - 2\sqrt{2}\right)^2\right) \\ &= \tfrac{1}{4}\pi\big(35 - 24\sqrt{2}\big) \\ &\approx 83.16\%. \end{aligned}$$

② With r as large as possible:

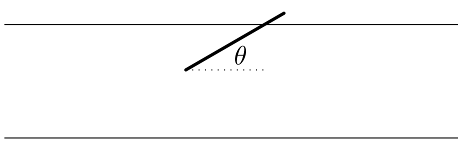


In this case, $r = 1/4$ and $R = 9/32$. This gives

$$\begin{aligned} A_{\text{total}} &= \pi\left(\big(\tfrac{9}{32}\big)^2 + 2\big(\tfrac{1}{4}\big)^2\right) \\ &= \tfrac{209}{1024}\pi \\ &\approx 64.12\%. \end{aligned}$$

Hence, the optimal arrangement is with $R = 1/2$, covering a little over 83% of the area.

4975. The centre of the needle must land on one of the floorboards. Consider this floorboard as having height $y \in [0, 1]$. So, the centre of the needle lands at y, which is distributed uniformly over $[0, 1]$. The position in the x direction is not relevant. Let the needle have angle of inclination to the x axis θ, where θ is distributed uniformly over $[0, \pi/2)$.



The component of the length of the needle in y is $\sin\theta$. So, at angle θ, any y value within $\frac{1}{2}\sin\theta$ of either 0 or 1 will result in the needle crossing a crack. The possibility space has size 1, so the probability of this is $\sin\theta$.

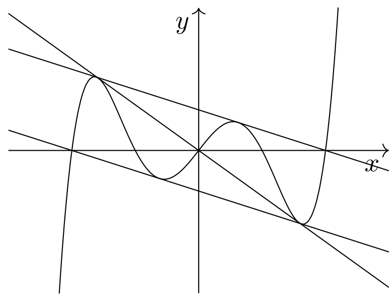
To calculate the total probability, we integrate this over all possible orientations $\theta \in [0, \pi/2)$, dividing by the width of the interval. This gives

$$\begin{aligned} p &= \frac{1}{\pi/2}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \\ &= \tfrac{2}{\pi}\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} \\ &= \tfrac{2}{\pi}, \text{ as required.} \end{aligned}$$

4976. (a) Factorising, the curve Q is

$$y = x(x-2)(x+2)(x-4)(x+4).$$

So, the curve is as shown below, with the three double tangents added:



It is clear visually that there can be no others. For proof, the following algebra serves.

(b) The steepest double tangent passes through the origin. This has equation $y = kx$. The equation for intersections between it and Q is

$$\begin{aligned} & x^5 - 20x^3 + 64x = kx \\ \implies & x^5 - 20x^3 + (64-k)x = 0 \\ \implies & x(x^4 - 20x^2 + (64-k)) = 0. \end{aligned}$$

For a double tangent, the quartic factor must be expressible as

$$\begin{aligned} x^4 - 20x^2 + 64 - k &\equiv (x-a)^2(x+a)^2 \\ &\equiv x^4 - 2a^2x^2 + a^4. \end{aligned}$$

Equating coefficients of x^2, we get $a = \sqrt{10}$. The constant terms then give $64 - k = a^4$, so $k = -36$.

The other tangents have equation $y = mx + c$. For intersections,

$$\begin{aligned} & x^5 - 20x^3 + 64x = mx + c \\ \implies & x^5 - 20x^3 + (64-m)x - c = 0. \end{aligned}$$

This must be expressible as

$$\begin{aligned} & x^5 - 20x^3 + (64-m)x - c \\ \equiv\ & (x+p)^2(x+q)^2(x+r) \\ \equiv\ & \big(x^2 + 2px + p^2\big)\big(x^2 + 2qx + q^2\big)(x+r). \end{aligned}$$

Equating coefficients,

$$\begin{aligned} & x^4 : 0 = 2p + 2q + r \implies 2(p+q) = -r, \\ & x^3 : -20 = \big(p^2 + 4pq + q^2\big) + 2r(p+q), \\ & x^2 : 0 = 2pq(p+q) + \big(p^2 + 4pq + q^2\big)r. \end{aligned}$$

We can simplify these with the substitution $a = p + q$, $b = 2pq$:

$$\begin{aligned} x^4 &: 2a = -r, \\ x^3 &: -20 = a^2 + b + 2ar, \\ x^2 &: 0 = ab + (a^2 + b)r. \end{aligned}$$

Eliminating r,

$$\begin{aligned} -20 = a^2 + b - 4a^2 &\implies 20 = 3a^2 - b, \\ 0 = ab - 2a^3 - 2ab &\implies 0 = a(2a^2 + b). \end{aligned}$$

In the latter, $a = 0$ corresponds to the tangent through the origin. Dividing by a, we have

$$\begin{aligned} 20 &= 3a^2 - b, \\ 0 &= 2a^2 + b. \end{aligned}$$

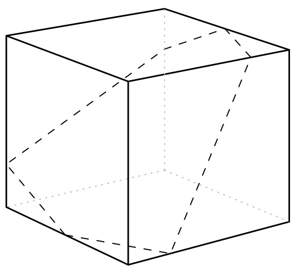
Adding gives $a = 2$ and $b = -8$. Solving for p and q yields $p, q = 1 \pm \sqrt{5}$, and then $r = 4$. So, the equation for intersections is

$$x^5 - 20x^3 + 80x - 64 = 0.$$

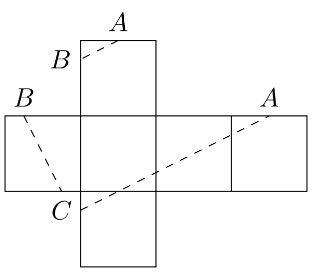
The gradient of the tangent is -16.

Collating this information, the double tangent through the origin has gradient -36 and the other two have gradient -16.

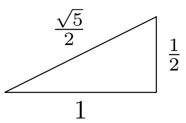
4977. On the original cube, the path is



Unwrapped to a flat net, this is



The path consists of a total of four copies of the hypotenuse of the triangle shown:



So, the total length of the loop is $2\sqrt{5}$.

4978. At height z, the cross-section of the region T is a triangle. Setting $y = 0$, its x width is given by $x = 1 - z$. By symmetry, the y width is the same. So, the area of the triangle at height z is given by

$$A_z = \tfrac{1}{2}(1 - z)^2.$$

To find the volume, we integrate this area from $z = 0$ to $z = 1$:

$$\begin{aligned} V &= \int_{z=0}^{z=1} A_z \, dz \\ &= \int_0^1 \tfrac{1}{2}(1 - z)^2 \, dz \\ &= \Big[-\tfrac{1}{6}(1 - z)^3 \Big]_0^1 \\ &= \tfrac{1}{6}, \text{ as required.} \end{aligned}$$

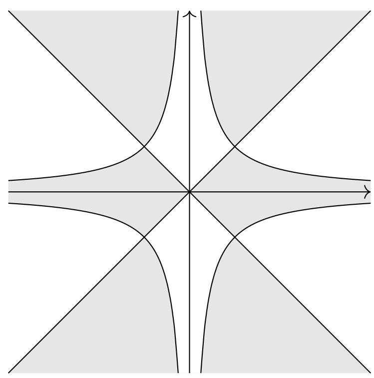
4979. Factorising, the inequality is

$$(x - y)(x + y)(xy - 1)(xy + 1) \le 0.$$

So, the boundary equations are

$$y = \pm x, \quad xy = \pm 1.$$

There are no repeated factors. So, crossing over any boundary equation changes the sign of the LHS. Hence, we have a chequerboard pattern bounded by the given lines, with every other region shaded. Testing $(1, 0)$, we see that the region containing the positive x axis satisfies the inequality. All of the other regions follow from there:



4980. The possibility space consists of the 4^5 ways of colouring the regions. There are four ways of colouring the central region. Having done this, for successful outcomes, three colours remain for the outside four.

① Type ABAC. There are 3 choices for the pair. Having chosen the pair, the number of orders is 4.

② Type ABAB. There are 3 choices for the two pairs, and then 2 orders, giving 6 outcomes.

The probability is

$$p = \frac{4(3 \times 4 + 3 \times 2)}{4^5} = \tfrac{9}{128}.$$

— Alternative Method —

Colour the central region red, wlog. Success then requires no red regions around the outside. This has probability

$$\left(\tfrac{3}{4}\right)^4 = \tfrac{81}{256}.$$

We now have four regions in a ring, each of which is coloured yellow, green or blue. Colour one of them yellow, wlog. For success, the two adjacent regions cannot be yellow. They can either be the same colour as each other or different colours:

① The probability that both adjacent regions are the same non-yellow colour is

$$2 \times \left(\tfrac{1}{3}\right)^2 = \tfrac{2}{9}.$$

Say both are blue: the remaining region can be yellow or green, with probability $\frac{2}{3}$.

② The probability that the adjacent regions are coloured one green, one blue is

$$2 \times \left(\tfrac{1}{3}\right)^2 = \tfrac{2}{9}.$$

Success then requires the last region to be yellow, with probability $\frac{1}{3}$.

So, the overall probability is

$$p = \tfrac{81}{256} \times \tfrac{2}{9}\left(\tfrac{2}{9} \times \tfrac{2}{3} + \tfrac{2}{9} \times \tfrac{1}{3}\right) = \tfrac{9}{128}.$$

4981. Consider the following expression:

$$\begin{aligned} &\cos(\theta-\phi) - \cos(\theta+\phi) \\ \equiv\ &\cos\theta\cos\phi + \sin\theta\sin\phi \\ &\qquad - (\cos\theta\cos\phi - \sin\theta\sin\phi) \\ \equiv\ &2\sin\theta\sin\phi. \end{aligned}$$

Setting $\theta = 5x$ and $\phi = 3x$, this gives

$$\sin 5x \sin 3x \equiv \tfrac{1}{2}(\cos 2x - \cos 8x).$$

We can now integrate:

$$\begin{aligned} &\int \sin 5x \sin 3x \, dx \\ =\ &\tfrac{1}{2}\int \cos 2x - \cos 8x \, dx \\ =\ &\tfrac{1}{4}\sin 2x - \tfrac{1}{16}\sin 8x + c. \end{aligned}$$

— Nota Bene —

The quickest way to solve this problem is with the following *product-to-sum identity*:

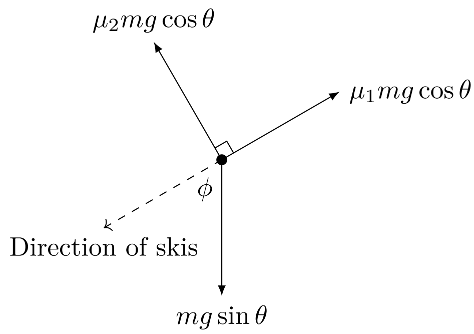
$$\sin a \sin b \equiv \frac{\cos(a-b) - \cos(a+b)}{2}.$$

This is not assumed knowledge in this book.

4982. Assume, for a contradiction, limiting friction in both components. The reaction is $mg\cos\theta$, so the maximal frictions are

$$\begin{aligned} F_1 &= \mu_1 mg\cos\theta, \\ F_2 &= \mu_2 mg\cos\theta. \end{aligned}$$

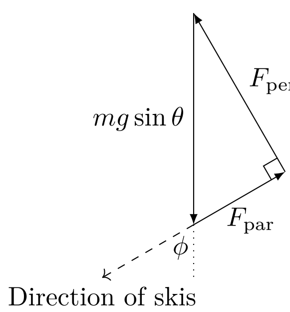
Ignoring the normal dimension, the 2D plane of the slope is as follows. Note that vertical in the diagram is not vertical in (x, y, z) space. Shown acting vertically down the page is the component of weight acting down the slope, which is $mg\sin\theta$. For visualisation: the background of the following diagram is the surface of the snow: the three forces *all act along the surface of the snow.*



For constant speed, the resultant friction must point up the slope. Its magnitude is

$$\begin{aligned} &mg\cos\theta\sqrt{\mu_1^2 + \mu_2^2} \\ >\ &mg\cos\theta\mu_2 \\ >\ &mg\cos\theta\tan\theta \\ =\ &mg\sin\theta. \end{aligned}$$

Therefore, the resultant force will act up the slope, and the skier will slow down. So, the choice of path in which both frictions are maximal is not steep enough. We have a choice: either reduce F_{par} or F_{per} below maximal. The triangle of forces is



To minimise ϕ (steepest course), we need to reduce F_{per} below maximal, while maintaining

$$F_{\text{par}} = \mu_1 mg\cos\theta.$$

In this case, the above triangle gives

$$\begin{aligned} &mg\sin\theta\cos\phi = \mu_1 mg\cos\theta \\ \Longrightarrow\ &\cos\phi = \mu_1\cot\theta, \text{ as required.} \end{aligned}$$

4983. The central hexagon is a ring of six regions, each bordering two neighbours. Successful colourings are of four types:

① Type ABABAB. There are 3 choices for the colours, then 2 orders, giving 6 outcomes.

② Type ABABAC. Since each appears differently, there are 3! choices for the colours. Then, having chosen, there are 2 location-sets for the As, and 3 locations for C. This gives $3! \cdot 2 \cdot 3 = 36$ outcomes.

③ Type ABCACB. There are 3 choices for the colour (A here) taking opposite spots. There are 3 location-pairs for this colour. There are then 2 orders for the remaining colours. This gives $3 \cdot 3 \cdot 2 = 18$.

④ Type ABCABC. There are $3! = 6$ orders.

Overall, there are $6+36+18+6 = 66$ possibilities.

Once the central hexagon has been coloured, the three remaining triangles are independent of each other. For every arrangement of the central hexagon, each individual outer triangle can be coloured independently in two ways. The total number of possibilities, therefore, is

$$66 \times 2^3 = 528.$$

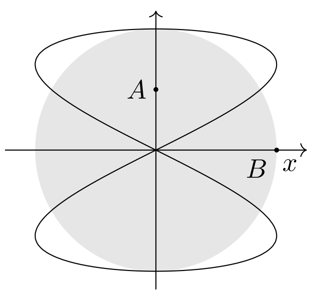
4984. Assume for a contradiction, that $k \in \mathbb{Z}$, $\sqrt{k} \notin \mathbb{Z}$, and $\sqrt{k} \in \mathbb{Q}$. Since $\sqrt{k}$ is rational, we can write it as $\sqrt{k} = p/q$, where $p, q \in \mathbb{N}$ have no common factors. Rearranging, we get

$$p^2 = kq^2.$$

Consider the prime factorisation of k. If any prime factor a appears to an odd power, then, since squares contain only even powers, the equation above will have different numbers of factors of a on its two sides. This is impossible. Hence, every prime factor of k must appear to an even power. But this means that k is a perfect square, which contradicts $\sqrt{k} \notin \mathbb{Z}$.

So, if $k \in \mathbb{Z}$ and $\sqrt{k} \notin \mathbb{Z}$, then $\sqrt{k} \notin \mathbb{Q}$. QED.

4985. The path P of the centre of the moving unit circle is shown below, as well as the circle C of radius 2. P is a Lissajous curve, consisting of independent sinusoidal motion in 2D, with frequencies 2 : 1. Its parametric equations are $x = 2\sin 2t$, $y = 2\sin t$.



We need to prove that every point shaded grey lies within 1 unit of the path P. To do this, we need only consider the points which could be maximally far from path P. There are, up to reflections, two such points, labelled A and B above.

Point A is certainly within 1 unit of path P, since a distance 1 down from $(0, 2)$ puts it at $(0, 1)$. The gradients at the origin are non-zero, so there are points either side of O sufficiently close to A.

Point B requires a calculation. We are looking for the point on P, in the positive quadrant, which is closest to B. For the gradient of the path P,

$$\begin{aligned}\frac{dx}{dt} &= 4\cos 2t,\\ \frac{dy}{dt} &= 2\cos t.\end{aligned}$$

Using the parametric differentiation formula,

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos t}{4\cos 2t}\\ &\equiv \frac{\cos t}{2\cos 2t}.\end{aligned}$$

The normal gradient, then, is given by

$$m_{\text{normal}} = \frac{-2\cos 2t}{\cos t}.$$

So, the equation of a generic normal is

$$y - 2\sin t = \frac{-2\cos 2t}{\cos t}(x - 2\sin 2t).$$

Setting this normal to pass through $(2, 0)$,

$$\begin{aligned}-2\sin t &= \frac{-2\cos 2t}{\cos t}(2 - 2\sin 2t)\\ \Longrightarrow\ 2\sin t\cos t &= 4\cos 2t(1 - \sin 2t)\\ \Longrightarrow\ \sin 2t &= 4\cos 2t(1 - \sin 2t).\end{aligned}$$

We solve numerically, giving $t = 0.4103$ radians to 4dp. We don't need this exactly, because we are only trying to find a point on the curve which is within unit distance of $(2, 0)$.

Substituting $t = 0.4103$ radians, the relevant point on P is $(1.46311..., 0.79776...)$. We can round the y value up and the x value down, guaranteeing that $(1.4631, 0.7978)$ is further from B than the curve P is from B. The distance between $(1.4631, 0.7978)$ and $(2, 0)$ is

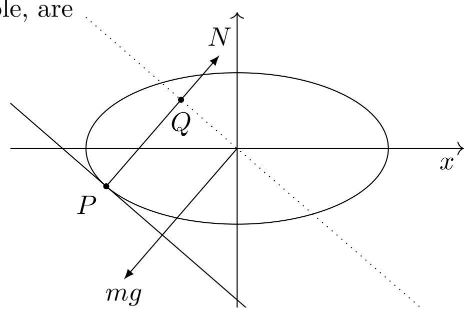
$$\begin{aligned}d &= \sqrt{0.5369^2 + 0.7978^2}\\ &= 0.96163...\\ &< 1.\end{aligned}$$

Hence, there are points on path P which are less than 1 unit distant from B. This proves that no points in the shaded circle C lie more than 1 unit away from path P. □

4986. Point P is $\left(-\frac{3\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. The gradient is

$$\frac{dy}{dx} = \frac{\sqrt{2}\cos t}{-\sqrt{6}\sin t}.$$

So, the normal gradient is $\sqrt{3}\tan t$. At P, this is $m = 1$. Hence, the equation of the normal at P is $y = x + \sqrt{2}$. Rotating the ground and gravity rather than the prism, the forces, not including the couple, are



Since the couple exerts no resultant force, we know that $N = mg$. Hence, N and mg must themselves form a couple, whose magnitude is equal to that applied externally.

To calculate the perpendicular distance between the forces, we solve for point Q. The equations are $y = x + \sqrt{2}$ and $y = -x$, so point Q is at $(-\sqrt{2}/2, \sqrt{2}/2)$. By Pythagoras, $|OQ| = 1$. So, the magnitude of the applied couple is

$$|\tau| = mg \text{ Nm}.$$

— Nota Bene —

The moment of a couple is the same around any point, which is why its magnitude is Fd, where F is the magnitude of each of the forces, and d is the perpendicular distance between them.

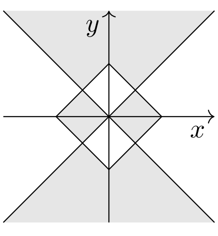
4987. We can rewrite x^2 as $|x|^2$ and y as $|y|^2$, giving

$$|x|^2 + |y| \le |y|^2 + |x|.$$

Since x only appears as $|x|$ and y only as $|y|$, the solution set is symmetrical in the x and y axes. So, consider the positive quadrant. The inequality is

$$\begin{aligned} & x^2 + y \le y^2 + x \\ \iff & x^2 - x \le y^2 - y \\ \iff & \left(x - \tfrac{1}{2}\right)^2 \le \left(y - \tfrac{1}{2}\right)^2. \end{aligned}$$

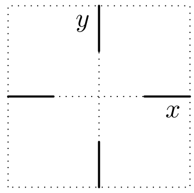
The boundary equations are $y = x$ and $x + y = 1$. We truncate these to the positive quadrant, and then mirror them in the axes. Checking regions, the solution set is



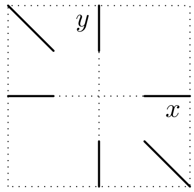
4988. We sketch the boundary equation. The magnitude of the indices means we can approximate the curve with a set of line segments. Factorising, it is

$$xy\left(x^{9999} + y^{9999}\right) = 1.$$

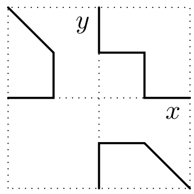
If e.g $|x|$ is greater than 1 but y isn't, then the heavy factor is huge, requiring e.g. y to be very close to zero. So, the first set of line segments is



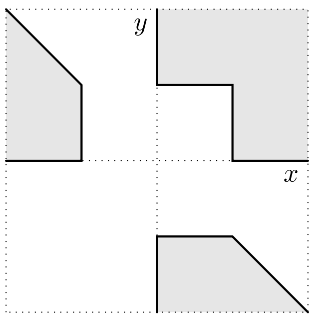
If both $|x|$ and $|y|$ are greater than 1, then they must be approximately negatives of one another to limit the heavy factor. This gives



There are also solution points with either x or y close to zero. These must join the existing lines. Noting that there are points close to $(1, 1)$, the full boundary graph is approximately



Testing points, the region satisfying the inequality is approximately



Counting up integer squares, 6 out of 16 are shaded. So, for a point chosen randomly in S,

$$\mathbb{P}\left(x^{10000}y + xy^{10000} \ge 1\right) \approx \tfrac{3}{8}, \text{ as required.}$$

4989. These three cubics have, respectively, one, two or no SPs. The number of SPs is not altered by stretches, translations or reflections. So, if a cubic graph can be transformed to one of these curves, then it cannot be transformed to any of the others.

Consider a cubic

$$y = ax^3 + bx^2 + cx + d.$$

A stretch by factor $\frac{1}{a}$ in the y direction converts this to the form

$$y = x^3 + \tfrac{b}{a}x^2 + \tfrac{c}{a}x + \tfrac{d}{a}.$$

We now complete the cube. This deals with the x^2 term, giving, for some constants p and q,

$$y = \left(x - \tfrac{b}{3a}\right)^3 + px + q.$$

We now translate by $\frac{b}{3a}\mathbf{i}$. This leaves, for some new constants r and s,

$$y = x^3 + rx + s.$$

Translating by $-s\mathbf{j}$, we have

$$y = x^3 + rx.$$

Consider the sign of r:

① If $r = 0$, then we have reached $y = x^3$.

② If $r > 0$, then we scale in the x direction by $1/\sqrt{r}$. This gives

$$\begin{aligned} y &= \left(\sqrt{r}x\right)^3 + r\left(\sqrt{r}x\right) \\ &= r^{\frac{3}{2}}(x^3 + x). \end{aligned}$$

Stretching by factor $r^{-\frac{3}{2}}$ in the y direction yields $y = x^3 + x$.

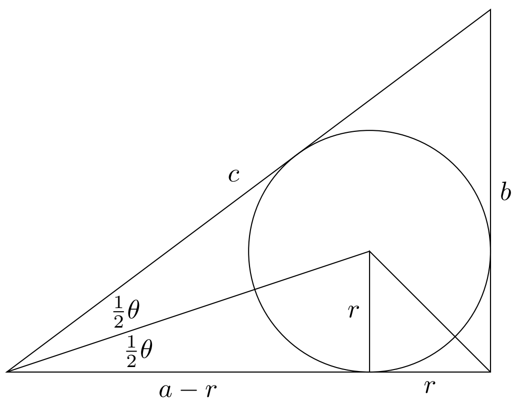
③ If $r < 0$, let $k = -r$. Applying the argument above then brings us to $y = x^3 - x$.

So, every cubic can be transformed to *at least* one of $y = x^3$, $y = x^3 + x$ or $y = x^3 - x$.

Therefore, classifying by SPs as described earlier, every cubic can be transformed to *exactly* one of $y = x^3$, $y = x^3 + x$ or $y = x^3 - x$. □

4990. According to the angle in a semicircle theorem, the hypotenuse is a diameter of the circumcircle, so $R = \frac{1}{2}c$.

To find r, set the triangle up as follows:



Since the incentre lies on the angle bisectors,

$$\tan\tfrac{1}{2}\theta = \frac{r}{a-r}.$$

Rearranging this,

$$r = \frac{a\tan\frac{1}{2}\theta}{1 + \tan\frac{1}{2}\theta}.$$

Using the given half-angle formula,

$$\begin{aligned} r &= \frac{a\frac{\sin\theta}{1+\cos\theta}}{1 + \frac{\sin\theta}{1+\cos\theta}} \\ &\equiv \frac{a\sin\theta}{\cos\theta + \sin\theta + 1}. \end{aligned}$$

Substituting the trig ratios,

$$\begin{aligned} r &= \frac{a \cdot \frac{b}{c}}{\frac{a}{c} + \frac{b}{c} + 1} \\ &\equiv \frac{ab}{a+b+c}. \end{aligned}$$

Lastly, we multiply by $R = \frac{1}{2}c$:

$$rR = \frac{abc}{2(a+b+c)}, \text{ as required.}$$

4991. (a) Let $u = \sec\theta + \tan\theta$. This gives

$$\begin{aligned} du &= \sec\theta\tan\theta + \sec^2\theta \\ \Longrightarrow \frac{du}{u} &= \sec\theta\, d\theta. \end{aligned}$$

Enacting the substitution,

$$\begin{aligned} &\int \frac{1}{u}\, du \\ =\ & \ln|u| + c \\ =\ & \ln|\sec\theta + \tan\theta| + c. \end{aligned}$$

(b) Let $x = \sec\theta$, so that $dx = \sec\theta\tan\theta\, d\theta$:

$$\begin{aligned} I &= \int \frac{2\sec^2\theta}{\sqrt{\sec^2\theta - 1}} \cdot \sec\theta\tan\theta\, d\theta \\ \Longrightarrow \tfrac{1}{2}I &= \int \sec^3\theta\, d\theta. \end{aligned}$$

For parts, let $u = \sec\theta$ and $v' = \sec^2\theta$, so that $u' = \sec\theta\tan\theta$ and $v = \tan\theta$. The parts formula gives

$$\begin{aligned} \tfrac{1}{2}I &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta\, d\theta \\ &= \sec\theta\tan\theta - \tfrac{1}{2}I + \int \sec\theta\, d\theta. \end{aligned}$$

Rearranging,

$$I = \sec\theta\tan\theta + \int \sec\theta\, d\theta.$$

Using the result of part (a),

$$\begin{aligned} I &= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + c \\ &= x\sqrt{x^2-1} + \ln\left|x + \sqrt{x^2-1}\right| + c. \end{aligned}$$

4992. Graphically, $x^2 + y^2 = k$ is a circle. We sketch the inequality. The first difference of two squares is

$$(xy + 1)(xy - 1).$$

The second difference of two squares is

$$(x^2 - y^2 + 2)(x^2 - y^2 - 2).$$

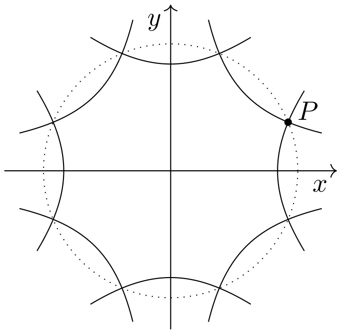
So, the boundary equation consists of the curves $xy \pm 1 = 0$, which are the standard hyperbolae asymptotic to the axes, and

$$x^2 - y^2 \pm 2 = 0.$$

These latter curves can be further factorised, to give $(x + y)(x - y) \pm 2 = 0$, which can be written

$$\tfrac{1}{\sqrt{2}}(x + y)\tfrac{1}{\sqrt{2}}(x - y) = 1.$$

Now, since $\frac{1}{\sqrt{2}}(x + y)$ and $\frac{1}{\sqrt{2}}(x - y)$ are variables of the same magnitude as x and y, the second two hyperbolae are identical to the first two, except they are asymptotic to $y = \pm x$, rather than the axes. This tells us that the boundary equation has rotational symmetry order 8 around the origin.



Since the origin doesn't satisfy the inequality, the region whose points satisfy the inequality may be reached by crossing exactly one boundary line from the central region. These are the regions, away from the origin, along the axes and $y = \pm x$.

Hence, if all points on a circle $x^2 + y^2 = k$ are to satisfy the inequality, then the circle must pass through the points of self-intersection. To find the radius at which this occurs, we can solve e.g. $xy = 1$ and $x^2 - y^2 - 2 = 0$. Substituting the latter into the former yields a biquadratic, which can be solved using the formula. The coordinates of the point marked P in the diagram above are

$$\left(\sqrt{1 + \sqrt{2}}, (\sqrt{2} - 1)\sqrt{1 + \sqrt{2}}\right).$$

The squared distance of P from O is

$$\begin{aligned} r^2 &= \left(1 + \sqrt{2}\right) + \left(\sqrt{2} - 1\right)^2\left(1 + \sqrt{2}\right) \\ &= \left(1 + \sqrt{2}\right)\left(1 + 2 - 2\sqrt{2} + 1\right) \\ &= \left(1 + \sqrt{2}\right)\left(4 - 2\sqrt{2}\right) \\ &= 2\sqrt{2}. \end{aligned}$$

Therefore, if a point is on the circle $x^2 + y^2 = 2\sqrt{2}$, it will remain entirely within regions satisfying the inequality. So, $k = 2\sqrt{2}$.

4993. If $\text{hcf}(a, b) = 1$, then a and b share no common factors. Consider the set

$$S = \{ax + by : x, y \in \mathbb{Z} \text{ and } ax + by > 0\}.$$

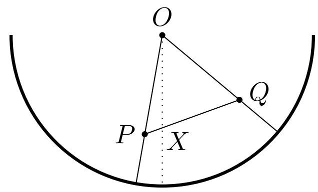
S is a set of positive integers. So, it must have a smallest element $s = ap + bq$. We need to show that $s = 1$. To do this, it suffices to show that s is a divisor of both a and b. Consider division of a by s, in the form $a = ns + r$, where r is a non-negative remainder $0 \le r < s$. We can rewrite this as

$$\begin{aligned} r &= a - ns \\ &= a - n(ap + bq) \\ &\equiv a(1 - np) - b(aq). \end{aligned}$$

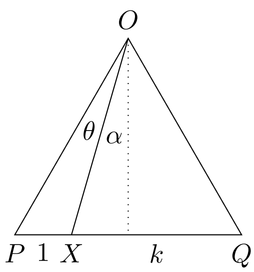
So, r is of the form $ax + by$, and must either be in S or be zero. But $r < s$, which is minimal, so $r \notin S$. So, r must be zero; s is a factor of a.

The same argument tells us that s is a factor of b. Hence, since $\text{hcf}(a, b) = 1$, s must be equal to 1. This gives us the result: if the highest common factor of $a, b \in \mathbb{N}$ is 1, then there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$. $\square$

4994. Consider the pair of spheres as a single object. It has three forces acting on it: two reactions acting radially, and the combined weight. Both reactions pass through the centre of the both. Hence, so must the weight, meaning that the centre of mass of the combined object must lie directly below O, on the intersection of the vertical and PQ. The centre of mass, marked X below, divides PQ in the ratio $1 : k$. The relevant diagram is



The problem is now solely geometric. Rotating the picture so that PQ lies horizontal, we have



The dotted height of the triangle is

$$h = \frac{k + 1}{2} \times \tan 60^\circ = \frac{\sqrt{3}(k + 1)}{2}.$$

Using the right-angled triangle containing α,

$$\begin{aligned}\tan\alpha &= \frac{\frac{k-1}{2}}{\frac{\sqrt{3}(k+1)}{2}}\\ &\equiv \frac{k-1}{\sqrt{3}(k+1)}.\end{aligned}$$

We know that $\theta = 30° - \alpha$. So,

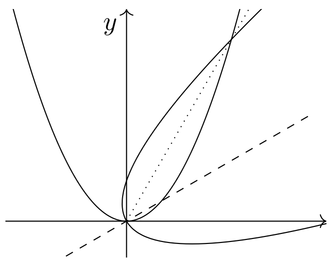
$$\begin{aligned}\tan\theta &= \tan(30° - \alpha)\\ &\equiv \frac{\tan 30° - \tan\alpha}{1+\tan 30°\tan\alpha}\\ &\equiv \frac{\frac{\sqrt{3}}{3} - \frac{k-1}{\sqrt{3}(k+1)}}{1+\frac{\sqrt{3}}{3}\cdot\frac{k-1}{\sqrt{3}(k+1)}}.\end{aligned}$$

Multiplying top and bottom by $\sqrt{3}(k+1)$,

$$\begin{aligned}\tan\theta &= \frac{k+1-(k-1)}{\sqrt{3}(k+1)+\sqrt{3}(k-1)}\\ &\equiv \frac{1}{k\sqrt{3}}.\end{aligned}$$

So, $\cot\theta = k\sqrt{3}$, as required.

4995. Sketch:



The dotted and dashed lines above are the images of the y axis under rotation by angles $\frac{1}{2}\theta$ and θ respectively. By symmetry, the curves meet on the dotted line, which has equation $y = x\cot\frac{1}{2}\theta$. Hence, the intersections of the curves are at

$$\begin{aligned}x^2 &= x\cot\tfrac{1}{2}\theta\\ \implies x &= 0, \cot\tfrac{1}{2}\theta.\end{aligned}$$

The area enclosed, then, is given by

$$\begin{aligned}A &= 2\int_0^{\cot\frac{1}{2}\theta} x\cot\tfrac{1}{2}\theta - x^2\,dx\\ &\equiv 2\Big[\tfrac{1}{2}x^2\cot\tfrac{1}{2}\theta - \tfrac{1}{3}x^3\Big]_0^{\cot\frac{1}{2}\theta}\\ &\equiv 2\Big(\tfrac{1}{2}\cot^3\tfrac{1}{2}\theta - \tfrac{1}{3}\cot^3\tfrac{1}{2}\theta\Big)\\ &\equiv \tfrac{1}{3}\cot^3\tfrac{1}{2}\theta.\end{aligned}$$

From here, a tan half-angle formula is simplest, though double-angle formulae can also be used in the other direction. The relevant formula is

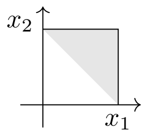
$$\tan\tfrac{1}{2}\theta \equiv \frac{1-\cos\theta}{\sin\theta}.$$

When reciprocated and cubed, this gives

$$\begin{aligned}A &= \tfrac{1}{3}\cot^3\tfrac{1}{2}\theta\\ &\equiv \frac{\sin^3\theta}{3(1-\cos\theta)^3}, \text{ as required.}\end{aligned}$$

4996. Each x_i in the tuple $(x_1, x_2, ..., x_n)$ can be thought of as position on one of n perpendicular axes. The possibility space is an n-dimensional hypercube of side length 1, whose (hyper)volume is 1. To find the probability, then, we need the volume of the relevant successful region.

With $n = 2$, this is obvious: the possibility space is a square, and half of it is successful:



For simplicity in calculation, we reflect this in the line $x_1 = \frac{1}{2}$, placing an acute vertex at the origin. This gives the area, and therefore probability, as

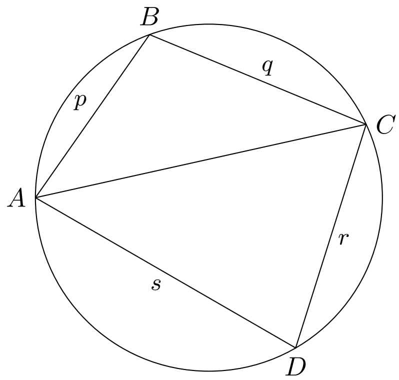
$$p_2 = \int_0^1 x_1\,dx_1 = \tfrac{1}{2}.$$

The 3D version has a pyramid as the success space. In that case, we can consider the pyramid as an integral of triangular slices. The triangular slices have area given by $\frac{1}{2}x_2^2$, where the factor of $\frac{1}{2}$ is p_2 and x_2^2 is an area scale factor. Hence,

$$\begin{aligned}p_3 &= \int_0^1 p_2 x_2^2\,dx_2\\ &= \Big[\tfrac{1}{3}p_2x_2^3\Big]_0^1\\ &= \tfrac{1}{3}p_2.\end{aligned}$$

The same process continues iteratively, leading to $p_n = \frac{1}{n}p_{n-1}$. Since $p_1 = 1$, $p_n = \frac{1}{n!}$, as required.

4997. We set up as follows, splitting the quadrilateral into two triangles with chord AC:



The total area H is given by

$$H = \tfrac{1}{2}pq\sin B + \tfrac{1}{2}rs\sin D.$$

B and D are opposite in a cyclic quadrilateral, so $\sin B = \sin D$. Hence,

$$2H = (pq+rs)\sin B.$$

Squaring, we can rearrange to

$$4H^2 = (pq + rs)^2 - (pq + rs)^2 \cos^2 B.$$

We now find an expression for $\cos^2 B$, using the cosine rule. Calculating the squared length $|AC|^2$ in both triangles,

$$p^2 + q^2 - 2pq \cos B = r^2 + s^2 - 2rs \cos D.$$

The cosines are related as $\cos D = -\cos B$, which means we can simplify to

$$(pq + rs) \cos B = \tfrac{1}{2}\big(p^2 + q^2 - r^2 - s^2\big).$$

Squaring gives

$$(pq + rs)^2 \cos^2 B = \tfrac{1}{4}\big(p^2 + q^2 - r^2 - s^2\big)^2.$$

Substituting into our formula for $4H^2$,

$$16H^2 = 4(pq + rs)^2 - \big(p^2 + q^2 - r^2 - s^2\big)^2.$$

The RHS is a difference of two squares:

$$\Big[2pq + 2rs + p^2 + q^2 - r^2 - s^2\Big]\Big[2pq + 2rs - ...\Big].$$

This simplifies to

$$\Big[(p + q)^2 - (r - s)^2\Big]\Big[(r + s)^2 - (p - q)^2\Big].$$

Each square bracket is a difference of two squares, so we can factorise again to get four symmetrical factors. We can write each of them in terms of the semiperimeter $S = \frac{1}{2}(p + q + r + s)$, e.g.

$$(p + q + r - s) = 2S - 2s = 2(S - s).$$

The same applies to the others brackets, so we have

$$16H^2 = 16(S - p)(S - q)(S - r)(S - s).$$

Dividing by 16 and taking the square root gives

$$H = \sqrt{(S - p)(S - q)(S - r)(S - s)}.$$

This is Brahmagupta's formula.

4998. SET-UP

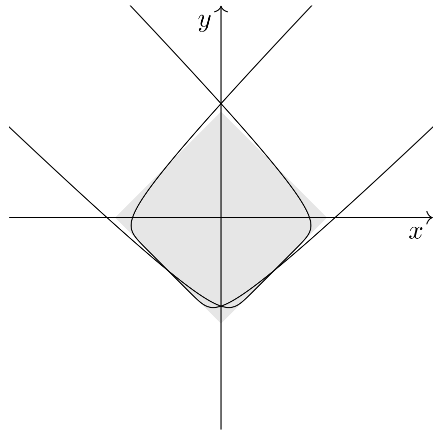
The inequalities given are phrased in terms of the variables $x + y$ and $x - y$. So, we rewrite, defining

$$X = x + y,$$
$$Y = x - y.$$

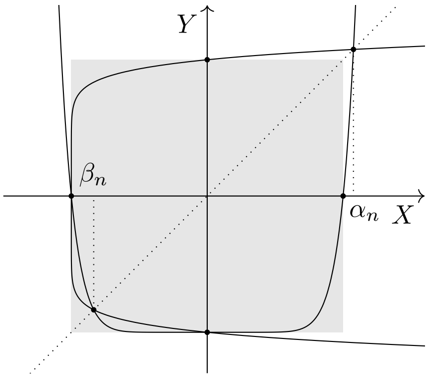
The boundary equations are now

$$Y = X^{2n} - 1$$
$$X = Y^{2n} - 1.$$

These are polynomial curves of even degree, at 45° to the (x, y) axes, as shown below. The grey square in the diagram has vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. It has area 2.



From this point, we transfer over to (X, Y) axes. The relevant square now has vertices $(\pm 1, \pm 1)$. In this coordinate system, it has area 4. The task is to show that the relevant region in (X, Y) space does indeed tend towards this square:



INTERSECTIONS

The axes intercepts are all at ± 1. The equation for intersections with $Y = X$ is

$$X^{2n} - X - 1 = 0.$$

From our sketch, we know that, for all $n \in \mathbb{N}$, this equation has exactly two roots α_n, β_n. We begin with the positive root α_n, which is a little over 1.

BOUNDS ON α_n

The first task is to show that

$$\alpha_n \in \Big(1, 1 + \tfrac{1}{2n-1}\Big).$$

For this, we use a sign change method. Defining $\mathrm{f}(X) = X + 1 - X^{2n}$, we know that $\mathrm{f}(1) = 1 > 0$. Then, by the binomial expansion,

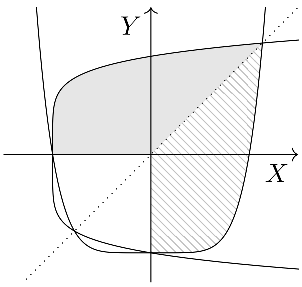
$$\begin{aligned} \mathrm{f}(1 + k) &= (1 + k) + 1 - (1 + k)^{2n} \\ &\equiv 2 + k - \big(1 + 2nk + {}^{2n}\mathrm{C}_2 k^2 + ... + k^n\big) \\ &\equiv 1 - (2n - 1)k - {}^{2n}\mathrm{C}_2 k^2 - ... - k^n. \end{aligned}$$

With k positive, the terms in k are all negative. Hence, if we set $k = \frac{1}{2n-1}$, then the first non-constant term is equal to -1, making the entire sum negative. We have our sign change. So,

$$\begin{aligned} \alpha &\in \Big(1, 1 + \tfrac{1}{2n-1}\Big) \\ &\equiv \Big(1, \tfrac{2n}{2n-1}\Big). \end{aligned}$$

AREA IN QUADRANTS 1, 2, 4

We can now find the limit of the following area:



The total area shaded above is given by twice the area of the hatched region. This we can calculate with a single integral:

$$\begin{aligned} A &= 2\int_0^{\alpha_n} X + 1 - X^{2n}\,dX \\ &\equiv 2\Big[\tfrac{1}{2}X^2 + X - \tfrac{1}{2n+1}X^{2n+1}\Big]_0^{\alpha_n} \\ &\equiv \alpha_n^2 + 2\alpha_n - \tfrac{2}{2n+1}\alpha_n^{2n+1}. \end{aligned}$$

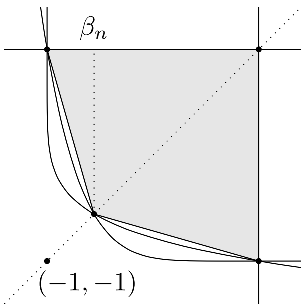
Substituting our interval, we can put bounds on this area. Setting $\alpha_n = 1$, its lower bound is clearly 3, since the last term vanishes as $n \to \infty$. Its upper bound, taking $\alpha_n = \frac{2n}{2n-1}$, is

$$\left(\tfrac{2n}{2n-1}\right)^2 + \tfrac{4n}{2n-1} - \tfrac{1}{2n+1}\left(\tfrac{2n}{2n-1}\right)^{2n+1}.$$

In this case, the last term is of the form $p(n)/q(n)$, where p has order $2n+1$ and q has order $2n+2$. Hence, in the limit, it must also vanish. Combined, the first two terms again tend to 3. Hence, by the squeeze theorem, the shaded area tends to 3.

AREA IN QUADRANT 3

We can now address the third quadrant, and the negative root β_n. Due to the positive curvature of polynomials of the form x^{2n}, the relevant area is larger than that of the following kite:



Hence, we need only show that $\beta_n \to -1$, as that will send the vertex on $Y = X$ to point $(-1,-1)$, and at that limit the kite is a square with area 1. So, we need to show that the negative root of $X^{2n} - X - 1 = 0$ can be set arbitrarily close to $X = -1$ by a suitably large choice of n.

Let δ be small and positive. Then, we look for a root in $(-1, -1+\delta)$, by a sign change method. We know $\mathrm{f}(-1) = 1 > 0$, so we need to show that $\mathrm{f}(-1+\delta) < 0$. This is

$$\begin{aligned} &(-1+\delta)^{2n} - (-1+\delta) - 1 < 0 \\ \iff &(-1+\delta)^{2n} < \delta. \end{aligned}$$

In the above, $(-1+\delta) \in (-1, 0)$. Hence, for any δ, $(-1+\delta)^{2n}$ can be rendered arbitrarily small by a large enough choice of n. In particular, it can be rendered smaller than δ, which puts the root β_n in the interval $(-1, -1+\delta)$. Hence, $\beta_n \to -1$ as $n \to \infty$. So, the remaining area tends to 1.

CONCLUSION

Overall, the limit in (X, Y) space is $3 + 1 = 4$. Translating this back into (x, y) space,

$$\lim_{n\to\infty} A_n = 2, \text{ as required.}$$

4999. APPROACH

If (a, b) is $(0, 0)$, the result is trivial. So, we need to prove that a change in the position of (a, b) causes no overall change in the shaded area. We prove this for changes in x. The same argument then holds for changes in y.

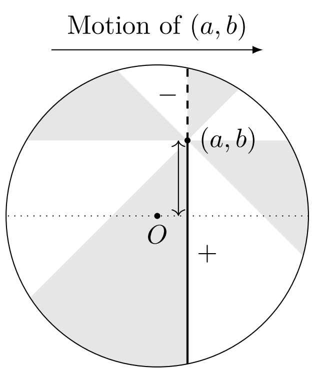
Consider translation of (a, b) to $(a + \delta x, b)$. As (a, b) moves, area is generated and destroyed at the straight edges of the shaded regions, in thin strips. Throughout the following argument, we work in the limit $\delta x \to 0$, in which those strips are line segments. The task is to sum the various contributions geometrically.

HORIZONTAL CHORD

Since we are moving horizontally, this chord does not contribute.

VERTICAL CHORD

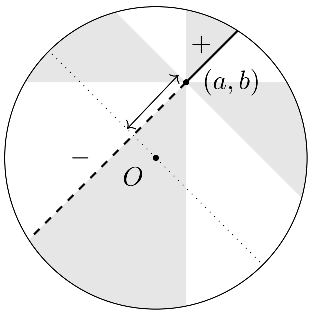
Consider the vertical chord through point (a, b). This has a lower section (shown solid below) and an upper section (shown dashed). As (a, b) moves in the positive x direction, the lower solid section contributes positively, generating new shaded area, while the upper section contributes negatively:



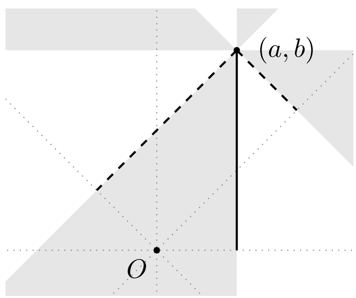
Now, the chord itself is symmetrical in $y = 0$, so the difference between the lengths of the sections is $2b$. This is twice the length marked above. This length can therefore be taken to encode (up to a universal factor of 2) the rate of change of area.

OBLIQUE CHORD

Consider an oblique chord. The calculation is very similar. This time, the signs are reversed, and the relevant direction, marked below, is oblique. It is the vertical component of this length that encodes the rate of change of area.



The same applies for the other oblique chord. We can now represent the entire calculation visually. The rate of change of area is given by the vertical components of the line segments below, with the vertical chord contributing positively and the two oblique chords contributing negatively:



The geometry of rectangles tells us that positive and negative contributions cancel exactly. So, the rate of change of area under translation in x, and likewise under translation in y, is zero. Hence, since $(0, 0)$ gives exactly half of the circle shaded, so must every other point (a, b). □

5000. CIRCLE AND PARABOLA

We begin with the circle and the parabola. Since the centre of the circle is $(-3, 0)$, we need to find a normal to the parabola passing through that point. A generic normal to the parabola is

$$y = -\frac{x}{2a} + a^2 + \tfrac{1}{2}.$$

Substituting $(-3, 0)$ gives

$$\begin{aligned} 0 &= \frac{3}{2a} + a^2 + \tfrac{1}{2} \\ \implies a &= -1. \end{aligned}$$

So, $(-1, 1)$ is the point on the parabola closest to the circle. The distance to the centre is $\sqrt{5}$. The radius of the circle is $\sqrt{6 - 2\sqrt{5}} = \sqrt{5} - 1$. Hence, the distance between circle and parabola is 1.

PARABOLA AND QUARTIC

Next, the parabola and the quartic. The parabola has a global minimum at $x = 0$. Differentiating the quartic,

$$\frac{dy}{dx} = -\frac{16(x^3 + 6x^2 + 9x)}{27(1 + \sqrt{5})}.$$

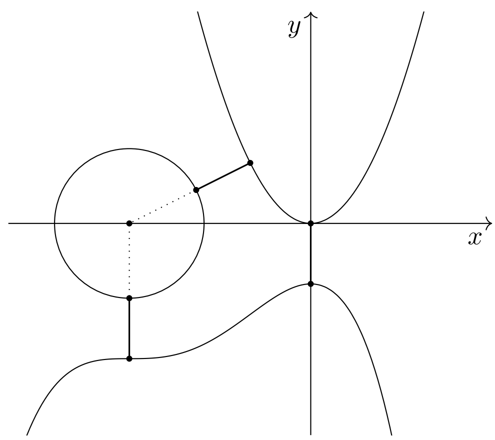
The curve is stationary when

$$\begin{aligned} & x^3 + 6x^2 + 9x = 0 \\ \implies & x(x+3)^2 = 0. \end{aligned}$$

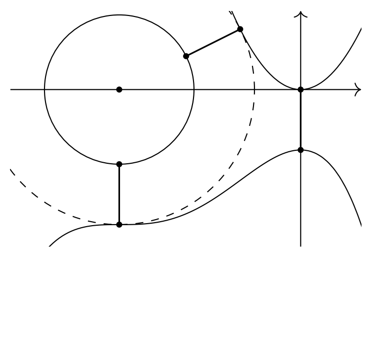
Since the derivative has a single root at $x = 0$ and a double root at $x = -3$, the curve has a point of inflection at $(-3, -\sqrt{5})$, and a local (and therefore global) maximum at $(0, -1)$. Hence, the shortest path between the quartic and the parabola must be along the y axis. Again, the distance is 1.

CIRCLE AND QUARTIC

Lastly, we come to the circle and the quartic. A generic normal might be doable, but would yield brutal equations. Easier is to notice that the point of inflection at $(-3, -\sqrt{5})$ is directly below the centre of the circle at $(-3, 0)$. Since the radius is $\sqrt{5} - 1$, the distance here is again 1. This looks like the shortest path.



It remains to prove rigorously that no other point on the quartic is closer to the circle. This means showing that a new circle of radius $\sqrt{5}$ and centre $(-3, 0)$ only intersects the quartic at $(-3, -\sqrt{5})$. The result we are looking for is evident, although as yet unproved, in the diagram below.



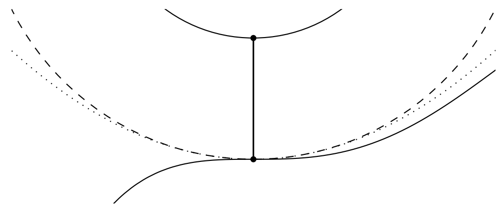
The equation of the new circle is $(x+3)^2 + y^2 = 5$. Again, if we brute force this, the equations look like getting heavy: a quartic and a circle don't combine nicely. Instead, we use an intermediary: a parabola, which will combine easily with each. We can find the parabola that best approximates the circle at $x = -3$ by considering the second derivative. Using the lower half of the circle:

$$\begin{aligned} y &= -\sqrt{5-(x+3)^2} \\ \implies \frac{d^2y}{dx^2} &= \frac{5}{(-(x+3)^2+5)^{\frac{3}{2}}}. \end{aligned}$$

Evaluating at $x = 3$ gives $1/\sqrt{5}$. The intermediary parabola has equation $y = k(x+3)^2 - \sqrt{5}$. Hence, $y'' = 2k$. Therefore, we require $k = 1/2\sqrt{5}$. So, our intermediary parabola has equation

$$y = \tfrac{1}{2\sqrt{5}}(x+3)^2 - \sqrt{5}.$$

This is shown dotted below, between the solid quartic and the dashed circle:



This parabola, by construction, stays at or below the circle. We need to show that it stays at or above the quartic. Solving for intersections, we take out the double factor at the point of tangency $x = -3$, giving

$$\begin{aligned} &27\left(1+\sqrt{5}\right)\left(\tfrac{1}{2\sqrt{5}}(x+3)^2 - \sqrt{5} + 1\right) \\ &\qquad + 4x^4 + 32x^3 + 72x^2 = 0 \\ \implies &\tfrac{1}{10}(x+3)^2\left(40x^2 + 80x + 15 + 27\sqrt{5}\right) = 0. \end{aligned}$$

The discriminant of the quadratic factor is

$$\begin{aligned} \Delta &= 80^2 - 4 \cdot 40\left(15 + 27\sqrt{5}\right) \\ &= 4000 - 4320\sqrt{5} < 0. \end{aligned}$$

This implies that the only intersection between the intermediary parabola and the quartic is the point of tangency at $x = -3$. Hence, the shortest path between circle and quartic is at that point.

CONCLUSION

The three curves are equidistant. QED.

——— END OF VOLUME V ———